

Axiomatic foundation of the n -dimensional Möbius geometry

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1. Introduction

G. EWALD [5] gave a simple system of axioms for the Möbius plane. In that plane there are only two fundamental concepts: the fundamental elements are circles and the fundamental relation is the relation of orthogonality of two circles. Another system of axioms for the Möbius plane was given in [8]. The analogous axiomatic construction of the n -dimensional Möbius space will be given here. Our axiomatic theory has only one set of fundamental elements, one fundamental relation and four axioms. Another system of axioms for the n -dimensional Möbius space was given by H. MÄURER [6].

2. Axioms

Let S be any nonempty set of elements which we call hyperspheres and let " $\perp$ " be a binary relation on it. If for hyperspheres a, b the relation $a \perp b$ is valid, then we say that the hypersphere a is orthogonal to the hypersphere b . Let n be any natural number.

Definition 1. An ordered $(n+2)$ -tuple $\mathfrak{A} = (a_1, a_2, \dots, a_{n+2})$ of hyperspheres is called an $(n+1)$ -simplexoid iff there is an ordered $(n+2)$ -tuple $\mathfrak{B} = (b_1, b_2, \dots, b_{n+2})$ of hyperspheres such that

$$(1) \quad (\forall \alpha, \beta \in \{1, 2, \dots, n+2\}) [a_\alpha \perp b_\beta \Leftrightarrow \alpha \neq \beta].$$

The notion of simplexoid is due to M. ESSER [4].

From Definition 1 it follows immediately

Theorem 1. If $(a_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid and $\alpha_1, \alpha_2, \dots, \alpha_{n+2}$ is any permutation of the indices $1, 2, \dots, n+2$, then $(a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_{n+2}})$ is an $(n+1)$ -simplexoid too.

Theorem 2. If $(a_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid, then $a_1, a_2, \dots, a_{n+2}$ are different hyperspheres.

PROOF. Let $\alpha, \beta \in \{1, 2, \dots, n+2\}, \alpha \neq \beta$. By Definition 1 there is an ordered $(n+2)$ -tuple $(b_1, b_2, \dots, b_{n+2})$ of hyperspheres such that the relation (1) holds. Then $a_\beta \perp b_\alpha$. If $a_\alpha = a_\beta$ were valid, then $a_\alpha \perp b_\alpha$ would follow, which is impossible. Hence $a_\alpha \neq a_\beta$.

Definition 2. A set $B \subseteq S$ of hyperspheres is called an independent set with respect to the relation of orthogonality or shortly $\perp$ -independent set iff there is an $(n+1)$ -simplexoid $(a_1, a_2, \dots, a_{n+2})$ such that $B \subseteq \{a_1, a_2, \dots, a_{n+2}\}$.

From this definition follows

Theorem 3. Every subset of an $\perp$ -independent set of hyperspheres is an $\perp$ -independent set.

Definition 3. We say that the set $A \subseteq S$ of hyperspheres is orthogonal to the set $B \subseteq S$ of hyperspheres and write $A \perp B$ iff $a \perp b$ for $\forall a \in A$ and $\forall b \in B$. In particular if $B = \{b\}$ is a singleton we say that the set A of hyperspheres is orthogonal to the hypersphere b and write $A \perp b$, and if $A = \{a\}$ is a singleton we say that the hypersphere a is orthogonal to the set B of hyperspheres and write $a \perp B$.

Definition 4. The structure $(S, \perp)$ is called an n -dimensional Möbius space iff:

- S1. $(\forall a, b \in S)[a \perp b \Rightarrow b \perp a]$.
- S2. $(\forall a_1, a_2, \dots, a_{n+1} \in S)(\exists b \in S)\{a_1, a_2, \dots, a_{n+1}\} \perp b$.
- S3. $(\exists a_1, a_2, \dots, a_{n+2} \in S)[b \in S \Rightarrow \neg (\{a_1, a_2, \dots, a_{n+2}\} \perp b)]$.
- S4. If $A = \{a_1, a_2, \dots, a_{n+1}\}$ is an $\perp$ -independent then

$$(\forall b, c \in S)[A \perp \{b, c\} \Rightarrow b = c].$$

Theorem 4. If $(a_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid, then there is one and only one ordered $(n+2)$ -tuple $\mathfrak{B} = (b_1, b_2, \dots, b_{n+2})$ of hyperspheres such that the relation (1) is valid. Then $\mathfrak{B}$ is also an $(n+1)$ -simplexoid.

PROOF. Let $(a_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid. By Definition 1 there is an ordered $(n+2)$ -tuple $\mathfrak{B} = (b_1, b_2, \dots, b_{n+2})$ of hyperspheres such that (1) holds. But, by S1, from (1) it follows

$$(\forall \alpha, \beta \in \{1, 2, \dots, n+2\})[b_\beta \perp a_\alpha \Leftrightarrow \alpha \neq \beta],$$

and by Definition 1 $\mathfrak{B}$ is an $(n+1)$ -simplexoid. Suppose that there is an $(n+1)$ -simplexoid $\mathfrak{B}' = (b'_1, b'_2, \dots, b'_{n+2})$ such that

$$(2) \quad (\forall \alpha, \beta \in \{1, 2, \dots, n+2\})[a_\alpha \perp b'_\beta \Leftrightarrow \alpha \neq \beta].$$

From (1) and (2) it follows

$$(3) \quad (\forall \alpha \in \{1, 2, \dots, n+2\})\{a_1, \dots, a_{\alpha-1}, a_{\alpha+1}, \dots, a_{n+2}\} \perp \{b_\alpha, b'_\alpha\}.$$

On the other hand, by Definition 2 we get that

$$(4) \quad (\forall \alpha \in \{1, 2, \dots, n+2\})\{a_1, \dots, a_{\alpha-1}, a_{\alpha+1}, \dots, a_{n+2}\} \text{ is an } \perp\text{-independent set.}$$

By S4 from (3) and (4) it follows

$$(\forall \alpha \in \{1, 2, \dots, n+2\}) b_\alpha = b'_\alpha,$$

i.e. $\mathfrak{B} = \mathfrak{B}'$, and the theorem is proved.

Definition 5. Two $(n+1)$ -simplexoides $\mathfrak{A} = (a_1, a_2, \dots, a_{n+2})$, $\mathfrak{B} = (b_1, b_2, \dots, b_{n+2})$ will be called dual $(n+1)$ -simplexoides and will be written $\mathfrak{A} = d(\mathfrak{B})$ or $\mathfrak{B} = d(\mathfrak{A})$ iff the relation (1) holds.

From Definition 1 and S1 follows

Theorem 5. $(\forall A, B \subseteq S)[A \perp B \Rightarrow B \perp A]$; $(\forall a \in S)(\forall B \subseteq S)[a \perp B \Leftrightarrow B \perp a]$.

3. $\perp$ -independent sets

Theorem 6. The set $A = \{a_1, a_2, \dots, a_{n+2}\}$ of hyperspheres is an $\perp$ -independent set iff there is no hypersphere b such that $A \perp b$.

PROOF. Let $A = \{a_1, a_2, \dots, a_{n+2}\}$ be the given set of hyperspheres. Suppose that

$$(5) \quad b \in S \Rightarrow \neg (A \perp b).$$

By S2 there are hyperspheres $b_1, b_2, \dots, b_{n+2}$ such that

$$(\forall \alpha \in \{1, 2, \dots, n+2\}) \{a_1, \dots, a_{\alpha-1}, a_{\alpha+1}, \dots, a_{n+2}\} \perp b_\alpha.$$

If for some $\alpha \in \{1, 2, \dots, n+2\}$ it were $a_\alpha \perp b_\alpha$, then it should be $A \perp b_\alpha$, contrary to hypothesis (5). Therefore (1) holds, i.e. $\mathfrak{A} = (a_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid and hence A is an $\perp$ -independent set. Conversely, let A be an $\perp$ -independent set, i.e. let $\mathfrak{A} = (a_1, a_2, \dots, a_{n+2})$ be an $(n+1)$ -simplexoid. Furthermore let $\mathfrak{B} = d(\mathfrak{A})$, $\mathfrak{B} = (b_1, b_2, \dots, b_{n+2})$. By Definition 5 and S4 b_1 is the only hypersphere such that $\{a_2, \dots, a_{n+2}\} \perp b_1$. By Definition 5 $\neg (a_1 \perp b_1)$, and the relation (5) follows.

From S3 and Theorem 6 we get

Theorem 7. There are hyperspheres $a_1, a_2, \dots, a_{n+2}$ such that $\{a_1, a_2, \dots, a_{n+2}\}$ is an $\perp$ -independent set.

Theorem 8. If $(a_1, a_2, \dots, a_{n+2})$, $(b_1, b_2, \dots, b_{n+2})$ are dual $(n+1)$ -simplexoides and c_1 is a hypersphere such that $\neg (c_1 \perp b_1)$, then $(c_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid.

PROOF. By Definition 5 and S4 b_1 is the unique hypersphere such that $\{a_2, \dots, a_{n+2}\} \perp b_1$, and as is by the hypothesis $\neg (c_1 \perp b_1)$, it follows that

$$b \in S \Rightarrow \neg (\{c_1, a_2, \dots, a_{n+2}\} \perp b).$$

By Theorem 6 $\{c_1, a_2, \dots, a_{n+2}\}$ is an $\perp$ -independent set, i.e. $(c_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid.

Theorem 9. For $\forall c_1 \in S$, $\{c_1\}$ is an $\perp$ -independent set.

PROOF. By Theorem 7 there is an $(n+1)$ -simplexoid $\mathfrak{A}=(a_1, a_2, \dots, a_{n+2})$. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \dots, b_{n+2})$. Then $B=\{b_1, b_2, \dots, b_{n+2}\}$ is an $\perp$ -independent set, and from Theorem 6 follows

$$c \in S \Rightarrow \neg (c \perp B).$$

Therefore there is a hypersphere in B which is not orthogonal to c_1 . Because of the symmetry we may suppose $\neg (c_1 \perp b_1)$, and then, by Theorem 8 $(c_1, a_2, \dots, a_{n+2})$ is an $(n+1)$ -simplexoid, i.e. $\{c_1\}$ is an $\perp$ -independent set.

Theorem 10. *If c_1, c_2 are different hyperspheres, then $\{c_1, c_2\}$ is an $\perp$ -independent set.*

PROOF. By Theorem 9 $\{c_1\}$ is an $\perp$ -independent set, and there is an $(n+1)$ -simplexoid $\mathfrak{A}=(c_1, a_2, \dots, a_{n+2})$ with the hypersphere c_1 as one element. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \dots, b_{n+2})$. By Definition 5 and S4 c_1 is the unique hypersphere such that $c_1 \perp \{b_2, \dots, b_{n+2}\}$, and because of $c_1 \neq c_2$, there is at least one hypersphere in the set $\{b_2, \dots, b_{n+2}\}$ which is not orthogonal to c_2 . By symmetry we may assume that $\neg (c_2 \perp b_2)$, and then by Theorem 8 it follows that (c_1, c_2, a_3, a_{n+2}) is an $(n+1)$ -simplexoid, and therefore $\{c_1, c_2\}$ is an $\perp$ -independent set.

Theorem 11. *Let $m \in \{2, \dots, n+2\}$. If $\{c_1, \dots, c_{m-1}\} \subseteq S$ in an $\perp$ -independent set and c_m, d_m such hyperspheres that*

$$(\{c_1, \dots, c_{m-1}\} \perp d_m) \& \neg (c_m \perp d_m),$$

then $\{c_1, \dots, c_{m-1}, c_m\}$ is an $\perp$ -independent set.

PROOF. By the hypothesis there is an $(n+1)$ -simplexoid $\mathfrak{A}=(c_1, \dots, c_{m-1}, a_m, \dots, a_{n+2})$. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \dots, b_{n+2})$. By Theorem 6 it follows that there is no hypersphere which is orthogonal to the set $A=\{c_1, \dots, c_{m-1}, a_m, \dots, a_{n+2}\}$. Hence there is at least one hypersphere in A which is not orthogonal to d_m . But $\{c_1, \dots, c_{m-1}\} \perp d_m$, and by symmetry we can suppose that $\neg (a_m \perp d_m)$. Then by Theorem 8 $\mathfrak{D}=(b_1, \dots, b_{m-1}, d_m, b_{m+1}, \dots, b_{n+2})$ is an $(n+1)$ -simplexoid. Let $\mathfrak{E}=d(\mathfrak{D})$, $\mathfrak{E}=(e_1, e_2, \dots, e_{n+2})$ and $\alpha \in \{1, \dots, m-1\}$. By Definition 1 it follows for the $(n+1)$ -simplexoides $\mathfrak{D}, \mathfrak{E}$ that $e_\alpha \perp b_\beta$ for $\forall \beta \in \{1, 2, \dots, n+2\} \setminus \{m, \alpha\}$ and $e_\alpha \perp d_m$. By the hypothesis it is $c_\alpha \perp d_m$, and for the $(n+1)$ -simplexoides $\mathfrak{A}, \mathfrak{B}$ we have from Definition 1 $c_\alpha \perp b_\beta$ for $\forall \beta \in \{1, 2, \dots, n+2\} \setminus \{m, \alpha\}$. Hence

$$\{c_\alpha, e_\alpha\} \perp \{b_1, \dots, b_{\alpha-1}, b_{\alpha+1}, \dots, b_{m-1}, d_m, b_{m+1}, \dots, b_{n+2}\}.$$

But

$$\{b_1, \dots, b_{\alpha-1}, b_{\alpha+1}, \dots, b_{m-1}, d_m, b_{m+1}, \dots, b_{n+2}\}$$

is an $\perp$ -independent set, and by S4 it follows $e_\alpha = c_\alpha$ for $\forall \alpha \in \{1, \dots, m-1\}$, i.e. $\mathfrak{E}=(c_1, \dots, c_{m-1}, e_m, \dots, e_{n+2})$. By the hypothesis we have $\neg (c_m \perp d_m)$ and by Theorem 8 it follows that $(c_1, \dots, c_{m-1}, c_m, e_{m+1}, \dots, e_{n+2})$ is an $(n+1)$ -simplexoid. Hence $\{c_1, \dots, c_{m-1}, c_m\}$ is an $\perp$ -independent set.

Theorem 12. *Let $m \in \{2, \dots, n+2\}$. If $c_1, c_2, \dots, c_m, d_2, \dots, d_m$ are hyperspheres such that $c_\alpha \perp d_\beta$ for $\forall \alpha \in \{1, 2, \dots, m\}, \forall \beta \in \{\alpha+1, \dots, m\}$ and $\neg (c_\alpha \perp d_\alpha)$ for $\forall \alpha \in \{2, \dots, m\}$, then $\{c_1, c_2, \dots, c_m\}$ is an $\perp$ -independent set.*

PROOF. The theorem can be proved by induction. For $m=2$ the statement follows by Theorems 9 and 11. If $\{c_1, \dots, c_{m-1}\}$ is an $\perp$ -independent set, then by Theorem 11 $\{c_1, \dots, c_{m-1}, c_m\}$ is an $\perp$ -independent set too.

Theorem 13. Let $m \in \{1, 2, \dots, n+1\}$. If $\mathfrak{A}=(a_1, a_2, \dots, a_{n+2})$, $\mathfrak{B}=(b_1, b_2, \dots, b_{n+2})$ are dual $(n+1)$ -simplexoides and c, d hyperspheres such that $c \perp \{b_{m+1}, \dots, b_{n+2}\}$ and $\{a_1, \dots, a_m\} \perp d$, then $c \perp d$.

PROOF. If $m=1$, then $a_1 \perp d$ and $c \perp \{b_2, \dots, b_{n+2}\}$. By Definition 1 and S4 it follows $c=a_1$, hence $c \perp d$. If $m=n+1$, then $c \perp b_{n+2}$ and $\{a_1, \dots, a_{n+1}\} \perp d$. By Definition 1 and S4 it follows $d=b_{n+2}$, hence $c \perp d$. Let now $m \in \{2, \dots, n\}$. Then $c \perp \{b_{m+1}, \dots, b_{n+2}\}$ and $\{a_1, \dots, a_m\} \perp d$. By Definition 5 it follows that $a_\alpha \perp b_\beta$ for $\forall \alpha \in \{1, 2, \dots, n+1\}$ and $\forall \beta \in \{\alpha+1, \dots, n+1\}$, and $(\neg a_\alpha \perp b_\alpha)$ for $\forall \alpha \in \{2, \dots, n+1\}$. Let us suppose, contrary to the statement, that $\neg(c \perp d)$. Put

$$a'_1 = a_1, \dots, a'_m = a_m, \quad a'_{m+1} = c, \quad a'_{m+2} = a_{m+1}, \dots, a'_{n+2} = a_{n+1},$$

$$b'_1 = b_1, \dots, b'_m = b_m, \quad b'_{m+1} = d, \quad b'_{m+2} = b_{m+1}, \dots, b'_{n+2} = b_{n+1}.$$

Then $a'_\alpha \perp b'_\beta$ for $\forall \alpha \in \{1, 2, \dots, n+2\}$, $\forall \beta \in \{\alpha+1, \dots, n+2\}$ and $\neg(a'_\alpha \perp b'_\alpha)$ for $\forall \alpha \in \{2, \dots, n+2\}$. From Theorem 12 for $m=n+2$ it follows that $\{a'_1, a'_2, \dots, a'_{n+2}\} = \{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\}$ is an $\perp$ -independent set. Then by Theorem 6 it follows that there is no hypersphere which is orthogonal to the set $\{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\}$. But by Definition 5 it follows $\{a_1, \dots, a_m, a_{m+1}, \dots, a_{n+1}\} \perp b_{n+2}$ and by the hypothesis $c \perp b_{n+2}$. Hence $\{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\} \perp b_{n+2}$. This contradiction proves the statement of the theorem.

4. Subspaces

Definition 6. If P, Q are nonempty sets of hyperspheres, then we say that P is saturated with Q or that Q saturates P iff:

$$M1. P \perp Q,$$

$$M2. (\forall s \in S) [s \perp Q \Rightarrow s \in P].$$

An equivalent form of Definition 6 is obviously

Theorem 14. If P, Q are nonempty sets of hyperspheres, then P is saturated with Q iff P is the set of all hyperspheres p such that $p \perp Q$.

We shall say, by convention, that the set S of all hyperspheres is saturated with the empty set $\emptyset$, and conversely that $\emptyset$ is saturated with S , since here the conditions M1 and M2 are satisfied in a trivial way.

Definition 7. A set P of hyperspheres will be called the subspace of the space $(S, \perp)$ iff there is a set Q of hyperspheres such that P is saturated with Q .

Definition 8. Two subspaces P, Q of the space $(S, \perp)$ will be called complementary subspaces, and it will be written $P=C(Q)$ or $Q=C(P)$, iff P is saturated with Q and Q is saturated with P .

$$\text{Obviously } S=C(\emptyset), \emptyset=C(S).$$

From Definitions 6 and 8 it follows at once

Theorem 15. *The sets P, Q of hyperspheres are complementary subspaces of the space $(S, \perp)$ iff:*

$$\text{M1. } P \perp Q,$$

$$\text{M2. } (\forall s \in S)[s \perp Q \Rightarrow s \in P],$$

$$\text{M3. } (\forall s \in S)[s \perp P \Rightarrow s \in Q].$$

Analogously, from Theorem 14 and Definition 8 we get

Theorem 16. *The sets P, Q of hyperspheres are complementary subspaces of the space $(S, \perp)$ iff:*

$$\text{a) } (\forall s \in S)[s \perp Q \Leftrightarrow s \in P],$$

$$\text{b) } (\forall s \in S)[s \perp P \Leftrightarrow s \in Q].$$

It is obvious that the set P in Definition 6 is uniquely determined by the set Q (the converse is not generally valid), and hence each subspace of the space $(S, \perp)$ is uniquely determined by its complementary subspace.

Theorem 17. *If P_1, Q_1 and P_2, Q_2 are two pairs of complementary subspaces of the space $(S, \perp)$, then the relations $P_1 \subset P_2$ and $Q_1 \supset Q_2$ are equivalent.*

PROOF. It is obvious that from $P_1 \subset P_2$ follows $Q_1 \supseteq Q_2$. If it were $Q_1 = Q_2$ then it would follow $P_1 = P_2$. Hence $Q_1 \supset Q_2$. The converse is proved analogously.

Theorem 18. *If the set P of hyperspheres is saturated with the set Q of hyperspheres and Q is saturated with the set P_0 of hyperspheres then P, Q are complementary subspaces of the space $(S, \perp)$.*

PROOF. Since Q is saturated with P_0 and P is saturated with Q , it follows by Definition 6

$$(6) \quad P_0 \perp Q,$$

$$(7) \quad (\forall s \in S)[P_0 \perp s \Rightarrow s \in Q],$$

$$(8) \quad P \perp Q,$$

$$(9) \quad (\forall s \in S)[s \perp Q \Rightarrow s \in P].$$

Let $s \in P_0$. Then by (6) $s \perp Q$ and by (9) we get $s \in P$. Hence $P_0 \subseteq P$. Now, let $s \in S$. If $P \perp s$, then $P_0 \perp s$ and by (7) $s \in Q$. Hence

$$(10) \quad (\forall s \in S)[P \perp s \Rightarrow s \in Q].$$

From (8), (9) and (10) we get by Theorem 15 that P and Q are complementary subspaces of the space $(S, \perp)$.

5. Basic sets and dimension of a subspace

Definition 9. We say that a subspace P of the space $(S, \perp)$ is generated by the set P_0 of hyperspheres (or that the set P_0 is the set of generators of the subspace P), and write $P = \langle P_0 \rangle$, iff there is a set Q of hyperspheres such that P is saturated with Q and Q is saturated with P_0 .

From the proof of Theorem 18 it follows that $\langle P_0 \rangle = P$ implies $P_0 \subseteq P$.

Theorem 19. If P_0 is a nonempty set of hyperspheres, then there is a set $\bar{P}_0 \subseteq P_0$ of hyperspheres such that $\bar{P}_0$ is an $\perp$ -independent set and $\langle \bar{P}_0 \rangle = \langle P_0 \rangle$.

PROOF. There is a greatest number m , $m \in \{-1, 0, 1, \dots, n\}$, such that there exists an $(n+1)$ -simplexoid with $m+2$ elements from the set P_0 . Let $\mathfrak{P} = (p_1, p_2, \dots, p_{m+2})$ be such an $(n+1)$ -simplexoid and let $p_1, \dots, p_{m+2}$ be its elements from the set P_0 . Put $\bar{P}_0 = \{p_1, \dots, p_{m+2}\}$. Let $Q, \bar{Q}$ be the sets of hyperspheres saturated with $P_0, \bar{P}_0$ and $P, \bar{P}$ the sets of hyperspheres saturated with $Q, \bar{Q}$, respectively. Then, by Definition 9 it follows $P = \langle P_0 \rangle$, $\bar{P} = \langle \bar{P}_0 \rangle$. It is necessary to prove that $P = \bar{P}$. By Theorem 18 P, Q resp. $\bar{P}, \bar{Q}$ are complementary subspaces of the space $(S, \perp)$ and it is sufficient to prove $Q = \bar{Q}$. If $q \in Q$, then $P_0 \perp q$ and by $\bar{P}_0 \subseteq P_0$ it follows $\bar{P}_0 \perp q$, hence $q \in \bar{Q}$. Therefore we have $Q \subseteq \bar{Q}$. We prove $\bar{Q} \subseteq Q$. There are two cases:

1) $m = n$. The elements of the set $\bar{P}_0 = \{p_1, p_2, \dots, p_{n+2}\}$ constitute an $(n+1)$ -simplexoid $\mathfrak{P} = (p_1, p_2, \dots, p_{n+2})$ and by Theorem 6 it follows $\bar{Q} = \emptyset$, wherefrom by $Q \subseteq \bar{Q}$ we get $Q = \emptyset$, and so $Q = \bar{Q}$.

2) $m \in \{-1, 0, 1, \dots, n-1\}$. Let $q \in \bar{Q}$. Then $\bar{P}_0 \perp q$, i.e. $\{p_1, \dots, p_{m+2}\} \perp q$. If it were $q \notin Q$, a hypersphere $p' \in P_0$ would exist such that $\neg(p' \perp q)$ and then by Theorem 11 the subset $\{p_1, \dots, p_{m+2}, p'\} \subseteq P_0$ would be an $\perp$ -independent set, which contradicts the definition of the number m . Therefore $q \in Q$, and $\bar{Q} \subseteq Q$.

Definition 10. The set of generators P of the subspace P of the space $(S, \perp)$ will be called the basic set of that subspace iff P is an $\perp$ -independent set.

Theorem 20. If P, Q are nonempty complementary subspaces of the space $(S, \perp)$, then there is at least one pair of dual $(n+1)$ -simplexoides $\mathfrak{P} = (p_1, p_2, \dots, p_{n+2})$, $\mathfrak{Q} = (q_1, q_2, \dots, q_{n+2})$ of hyperspheres and a uniquely determined number $m \in \{-1, 0, 1, \dots, n-1\}$, such that $P = \langle \{p_1, \dots, p_{m+2}\} \rangle$, $Q = \langle \{q_{m+3}, \dots, q_{n+2}\} \rangle$.

PROOF. Obviously $\langle P \rangle = P$. By Theorem 19 there is a set $\bar{P} \subseteq P$ of hyperspheres such that $\langle \bar{P} \rangle = P$ and $\bar{P}$ is an $\perp$ -independent set. Let $p_1, \dots, p_{m+2}$, $m \in \{-1, 0, 1, \dots, n-1\}$ be the elements of the set $\bar{P}$. Then these hyperspheres are elements of an $(n+1)$ -simplexoid $\mathfrak{P} = (p_1, p_2, \dots, p_{n+2})$. Let $\mathfrak{Q} = d(\mathfrak{P})$, $\mathfrak{Q} = (q_1, q_2, \dots, q_{n+2})$. The set Q consists of all those hyperspheres that are orthogonal to the set P and by Definition 5 it follows $q_{m+3}, \dots, q_{n+2} \in Q$. Let $\bar{Q} = \{q_{m+3}, \dots, q_{n+2}\}$. Owing to $\bar{Q} \subseteq Q$,

$$(11) \quad p \perp \bar{Q}$$

is valid for $\forall p \in P$. Conversely, if for some $p \in S$ (11) is valid, then by Theorem 13 it follows $p \perp q$ for all hyperspheres q for which

$$(12) \quad \bar{P} \perp q.$$

But, since $\bar{P}$ is a set of generators of P , from (12) it follows $q \in Q$ and hence $p \perp q$ for $\forall q \in Q$. By Definition 6 $p \in P$. Therefore $\bar{Q}$ is a set of generators of Q . Suppose now that there are two pairs, identical or different, of dual $(n+1)$ -simplexoids of hyperspheres

$$\mathfrak{P} = (p_1, p_2, \dots, p_{n+2}), \quad \mathfrak{Q} = (q_1, q_2, \dots, q_{n+2})$$

and

$$\mathfrak{P}' = (p'_1, p'_2, \dots, p'_{n+2}), \quad \mathfrak{Q}' = (q'_1, q'_2, \dots, q'_{n+2})$$

and numbers $m, m' \in \{-1, 0, 1, \dots, n-1\}$ such that

$$\bar{P} = \{p_1, \dots, p_{m+2}\}, \quad \bar{P}' = \{p'_1, \dots, p'_{m'+2}\}$$

are the sets of generators of P and

$$\bar{Q} = \{q_{m+3}, \dots, q_{n+2}\}, \quad \bar{Q}' = \{q'_{m'+3}, \dots, q'_{n+2}\}$$

the sets of generators of Q . We show that $m = m'$. Suppose e.g. that $m < m'$. Then $m \in \{-1, 0, 1, \dots, n-2\}$. Therefore there are no more than $n+1$ hyperspheres among the hyperspheres

$$(13) \quad p_1, \dots, p_{m+2}, p'_{m'+3}, \dots, p'_{n+2}$$

and by S2 there is a hypersphere q which is orthogonal to each of the hyperspheres (13). The hypersphere q is orthogonal to the set $\bar{P}$, and as $\langle \bar{P} \rangle = P$, we get $q \in Q$. On the other hand from $\langle \bar{P}' \rangle = P$ it follows $\bar{P}' \subseteq P$ and hence $\bar{P}' \perp q$. Therefore $\{p'_1, \dots, p'_{m'+2}, p'_{m'+3}, \dots, p'_{n+2}\} \perp q$, which contradicts Theorem 6. According to this it cannot be $m < m'$, and by symmetry it cannot be $m' < m$ either. Thus $m = m'$.

By Theorem 20 and Definition 10 it follows that each nonempty subspace of the space $(S, \perp)$ has a basic set and that all basic sets have the same number of hyperspheres which is precisely $m+2$, $m \in \{-1, 0, 1, \dots, n\}$. For the basic set of the subspace S we can take each $\perp$ -independent subset of $n+2$ hyperspheres. This property enables us to give the following definition:

Definition 11. The number of elements of any basic set of a nonempty subspace P of the space $(S, \perp)$ diminished by 2 is called the dimension of P with respect to the relation " $\perp$ " or the $\perp$ -dimension of P . If the $\perp$ -dimension of P is m , then we write $\dim_{\perp} P = m$. In particular we define $\dim_{\perp} \emptyset = -2$.

Obviously $\dim_{\perp} S = n$.

By Theorem 20 and Definitions 10 and 11 it follows immediately

Theorem 21. If P and Q are complementary subspaces of the space $(S, \perp)$, then $\dim_{\perp} P + \dim_{\perp} Q = n - 2$.

Theorem 22. If P_1 and P_2 are subspaces of the space $(S, \perp)$ then from $P_1 \subset P_2$ it follows $\dim_{\perp} P_1 < \dim_{\perp} P_2$.

PROOF. Obviously from $P_1 \subset P_2$ it follows $\dim_{\perp} P_1 \leq \dim_{\perp} P_2$. Suppose that $\dim_{\perp} P_1 = \dim_{\perp} P_2$. Then the basic sets of the subspaces P_1 and P_2 have the same number of elements, and each basic set of P_1 is simultaneously the basic set of P_2 . As the subspace is uniquely determined by each of its sets of generators, it follows $P_1 = P_2$ in contradiction with the hypothesis $P_1 \subset P_2$. Therefore $\dim_{\perp} P_1 < \dim_{\perp} P_2$.

Theorem 23. *If P is a subspace of the space $(S, \perp)$, then the relation $\dim_{\perp} P = -1$ is valid iff the set P has exactly one element. For $\forall p_1 \in S$ the set $\{p_1\}$ is the subspace of the space $(S, \perp)$.*

PROOF. Let P be a subspace of the space $(S, \perp)$. If P contains exactly one element then by Theorem 9 it follows that $\dim_{\perp} P = -1$. Conversely, let $\dim_{\perp} P = -1$. If then subspace P would contain two different hyperspheres then by Theorem 10 it would be $\dim_{\perp} P \geq 0$ which is impossible. On the other hand from $P = \emptyset$ it would follow by Definition 11 that $\dim_{\perp} P = -2$. Therefore the subspace P contains exactly one hypersphere. Let now p_1 be any hypersphere, and let $P = \langle \{p_1\} \rangle$. Then $\dim_{\perp} P \geq -1$. Let $Q = C(P)$. By Theorem 9 there is an $(n+1)$ -simplexoid $\mathfrak{P} = (p_1, p_2, \dots, p_{n+2})$ with the hypersphere p_1 as one element. Let $\mathfrak{Q} = d(\mathfrak{P})$, $\mathfrak{Q} = (q_1, q_2, \dots, q_{n+2})$. By Definition 5 $p_1 \perp \{q_2, \dots, q_{n+2}\}$, and since the set Q is saturated (Theorem 18) with the set $\{p_1\}$, it follows that $q_2, \dots, q_{n+2} \in Q$. Since $\{q_2, \dots, q_{n+2}\}$ is an $\perp$ -independent set, $\dim_{\perp} Q \geq n-1$, wherefrom by Theorem 21 it follows that $\dim_{\perp} P \leq -1$. Therefore, together with $\dim_{\perp} P \geq -1$, we get $\dim_{\perp} P = -1$, and by the first part of the theorem it follows $P = \{p_1\}$.

6. Intersection and sum of subspaces

Theorem 24. *If P_1, Q_1 and P_2, Q_2 are two pairs of complementary subspaces of the space $(S, \perp)$ then $P_1 \cap P_2$ is a subspace too. If $Q = C(P_1 \cap P_2)$ then $Q = \langle Q_1 \cup Q_2 \rangle$.*

PROOF. Let Q be the set of hyperspheres saturated with the set $P_1 \cap P_2$. As the sets P_1, P_2, Q are saturated with the sets $Q_1, Q_2, P_1 \cap P_2$ respectively, so by Definition 6 it follows that

$$(14) \quad P_1 \perp Q_1,$$

$$(15) \quad (\forall s \in S)[s \perp Q_1 \Rightarrow s \in P_1],$$

$$(16) \quad P_2 \perp Q_2,$$

$$(17) \quad (\forall s \in S)[s \perp Q_2 \Rightarrow s \in P_2],$$

$$(18) \quad (P_1 \cap P_2) \perp Q,$$

$$(19) \quad (\forall s \in S)[(P_1 \cap P_2) \perp s \Rightarrow s \in Q].$$

Let $s \in P_1 \cap P_2$. Then from (14) and (16) follows $s \perp Q_1, s \perp Q_2$. Hence $s \perp (Q_1 \cup Q_2)$. Therefore we have

$$(20) \quad (P_1 \cap P_2) \perp (Q_1 \cup Q_2).$$

Let now $s \in S$. If $s \perp (Q_1 \cup Q_2)$, then $s \perp Q_1, s \perp Q_2$, and by (15) and (17) it follows $s \in P_1, s \in P_2$, i.e. $s \in P_1 \cap P_2$. Therefore we have

$$(21) \quad (\forall s \in S)[s \perp (Q_1 \cup Q_2) \Rightarrow s \in P_1 \cap P_2].$$

From (20) and (21) it follows by Definition 6 that $P_1 \cap P_2$ is saturated with the set $Q_1 \cup Q_2$. Thus by Definition 9 $\langle Q_1 \cup Q_2 \rangle = Q$ and hence $Q \supseteq Q_1 \cup Q_2$. Let now $s \in S$. If $s \perp Q$ then $s \perp (Q_1 \cup Q_2)$ and by (21) we get $s \in P_1 \cap P_2$. Therefore

$$(22) \quad (\forall s \in S)[s \perp Q \Rightarrow s \in P_1 \cap P_2].$$

From (18), (19) and (22) it follows by Theorem 15 that $P_1 \cap P_2, Q$ are complementary subspaces of the space $(S, \perp)$.

Definition 12. The subspace Q of the space $(S, \perp)$, which is complementary to the intersection $P_1 \cap P_2$ of two subspaces P_1, P_2 with $P_1 = C(Q_1), P_2 = C(Q_2)$, will be called the sum of the subspaces Q_1, Q_2 , and will be written $Q = Q_1 + Q_2$.

From Theorem 24 and Definition 12 it follows $Q_1 + Q_2 = \langle Q_1 \cup Q_2 \rangle$.

Theorem 25. If P_1, P_2 are two subspaces of the space $(S, \perp)$, then

$$C(P_1 \cap P_2) = C(P_1) + C(P_2), \quad C(P_1 + P_2) = C(P_1) \cap C(P_2).$$

PROOF. Let $Q_1 = C(P_1), Q_2 = C(P_2)$. By Definition 12 $P_1 \cap P_2, Q_1 + Q_2$ and $P_1 + P_2, Q_1 \cap Q_2$ are two pairs of complementary subspaces of the space $(S, \perp)$. Therefore

$$C(P_1 \cap P_2) = Q_1 + Q_2 = C(P_1) + C(P_2), \quad C(P_1 + P_2) = Q_1 \cap Q_2 = C(P_1) \cap C(P_2).$$

Theorem 26. If P is a subspace of the space $(S, \perp)$ with $\dim_{\perp} P = m, m \in \{-1, 0, 1, \dots, n-1\}$, and if $\{p_1, \dots, p_k\} \subseteq P, k \in \{1, \dots, m+1\}$ is an $\perp$ -independent set, then there are hyperspheres $p_{k+1}, \dots, p_{m+2} \in P$ such that $\{p_1, \dots, p_k, p_{k+1}, \dots, p_{m+2}\}$ is a basic set of the subspace P .

PROOF. The greatest number of elements of any $\perp$ -independent subset of the subspace P equals $\dim_{\perp} P + 2 = m + 2$. As $k < m + 2$, there is a hypersphere p_{k+1} in the set $P \setminus \{p_1, \dots, p_k\}$, such that $\{p_1, \dots, p_k, p_{k+1}\}$ is an $\perp$ -independent set. The statement of the theorem follows inductively.

Theorem 27. If P_1, P_2 are two subspaces of the space $(S, \perp)$ then

$$(23) \quad \dim_{\perp} (P_1 \cap P_2) + \dim_{\perp} (P_1 + P_2) = \dim_{\perp} P_1 + \dim_{\perp} P_2.$$

PROOF. Let $\dim_{\perp} P_1 = i, \dim_{\perp} P_2 = j, \dim_{\perp} (P_1 \cap P_2) = k$. Then obviously $k \leq i, k \leq j$. The greatest number of elements of any $\perp$ -independent subset of the subspaces $P_1, P_2, P_1 \cap P_2$ is $i+2, j+2, k+2$, respectively. First we prove the inequality

$$(24) \quad \dim_{\perp} (P_1 + P_2) \leq i + j - k.$$

We distinguish two cases:

1) $k \in \{-1, 0, 1, \dots, n\}$. Let $\{p_1, \dots, p_{k+2}\}$ be a basic set of the subspace $P_1 \cap P_2$. By Theorem 26 there are hyperspheres $p_{k+3}, \dots, p_{i+2} \in P_1$ such that $\{p_1, \dots, p_{i+2}\}$ is a basic set of the subspace P_1 , which in case $k = i$ is reduced to $\{p_1, \dots, p_{k+2}\}$. Similarly, there are hyperspheres $p_{i+3}, \dots, p_{i+j-k+2} \in P$ such that $\{p_1, \dots, p_{k+2}, p_{i+3}, \dots, p_{i+j-k+2}\}$ is a basic set of the subspace P_2 , which in case $k = j$ is reduced to $\{p_1, \dots, p_{k+2}\}$. If $Q_1 = C(P_1), Q_2 = C(P_2)$ then by Definition 12 $Q_1 \cap Q_2 = C(P_1 + P_2)$. Since $\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}\}$ is the basic set of the subspace P_1 , by Theorem 18 and Definition 9 it follows that for $\forall s \in S$ the relation $\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}\} \perp s$

is valid iff $s \in Q_1$. Similarly the relation $\{p_1, \dots, p_{k+2}, p_{i+3}, \dots, p_{i+j-k+2}\} \perp s$ is equivalent with $s \in Q_2$. Therefore for $\forall s \in S$ the relation

$$\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j-k+2}\} \perp s$$

is equivalent with $s \in Q_1 \cap Q_2$, and by Theorem 14 the set $Q_1 \cap Q_2$ is saturated with the set

$$\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j-k+2}\}.$$

Since on the other hand the set $P_1 + P_2$ is saturated with the set $Q_1 \cap Q_2$, by Definition 9 it follows that $\{p_1, \dots, p_{i+j-k+2}\}$ is the set of generators of the subspace $P_1 + P_2$. Since the $\perp$ -dimension of the subspace $P_1 + P_2$ is equal to the greatest number of the elements of any $\perp$ -independent subset of the set $\{p_1, \dots, p_{i+j-k+2}\}$ diminished by 2, relation (24) follows at once.

2) $k = -2$. Now $P_1 \cap P_2 = \emptyset$. If $\{p_1, \dots, p_{i+2}\}$ is a basic set of the subspace P_1 and $\{p_{i+3}, \dots, p_{i+j+4}\}$ is a basic set of the subspace P_2 , then similarly as in the case 1) we prove that $\{p_1, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j+4}\}$ is a set of generators of the subspace $P_1 + P_2$, and hence $\dim_{\perp}(P_1 + P_2) \cong i + j + 2$, wherefrom by $k = -2$ the relation (24) again follows.

Thus in both cases the relation (24) holds. By Definition 12 $Q_1 + Q_2 = C(P_1 \cap P_2)$, and by the use of Theorem 21 we get

$$(25) \quad \dim_{\perp}(Q_1 \cap Q_2) = n - 2 - \dim_{\perp}(P_1 + P_2),$$

$$(26) \quad \dim_{\perp}(Q_1 + Q_2) = n - 2 - \dim_{\perp}(P_1 \cap P_2),$$

$$(27) \quad \dim_{\perp} Q_1 = n - 2 - \dim_{\perp} P_1, \quad \dim_{\perp} Q_2 = n - 2 - \dim_{\perp} P_2.$$

From (24) by the definition of the numbers i, j, k it follows

$$(28) \quad \dim_{\perp}(P_1 \cap P_2) + \dim_{\perp}(P_1 + P_2) \cong \dim_{\perp} P_1 + \dim_{\perp} P_2.$$

Similarly

$$(29) \quad \dim_{\perp}(Q_1 \cap Q_2) + \dim_{\perp}(Q_1 + Q_2) \cong \dim_{\perp} Q_1 + \dim_{\perp} Q_2,$$

and hence by (25), (26) and (27) it follows

$$(29) \quad \dim_{\perp}(P_1 + P_2) + \dim_{\perp}(P_1 \cap P_2) \cong \dim_{\perp} P_1 + \dim_{\perp} P_2.$$

From (28) and (29) we get just the required relation (23).

Theorem 28. *If P and Q are subspaces of the space $(S, \perp)$ then*

$$\dim_{\perp}[P \cap C(Q)] = \dim_{\perp} P - \dim_{\perp} Q + \dim_{\perp}[C(P) \cap Q].$$

PROOF. By Theorem 21 $\dim_{\perp}[P \cap C(Q)] = n - 2 - \dim_{\perp} C[P \cap C(Q)]$, and as by Theorem 25 $C[P \cap C(Q)] = C(P) + C[C(Q)] = C(P) + Q$, so $\dim_{\perp}[P \cap C(Q)] = n - 2 - \dim_{\perp}[C(P) + Q]$. By Theorem 27 $\dim_{\perp}[C(P) + Q] = \dim_{\perp} C(P) + \dim_{\perp} Q - \dim_{\perp}[C(P) \cap Q]$, and hence $\dim_{\perp}[P \cap C(Q)] = n - 2 - \dim_{\perp} C(P) - \dim_{\perp} Q + \dim_{\perp}[C(P) \cap Q]$. By Theorem 21 $n - 2 - \dim_{\perp} C(P) = \dim_{\perp} P$, and therefore the required equation follows.

7. Induced Möbius geometries

Definition 13. A hypersphere s will be called a point-hypersphere or shortly a point iff $s \perp s$.

Definition 14. The subspace P of the space $(S, \perp)$ will be called regular iff $P \cap C(P) = \emptyset$ and singular otherwise.

Theorem 29. If a subspace P of the space $(S, \perp)$ contains no point P is a regular subspace.

PROOF. Suppose that P is a singular subspace. Then by Definition 14 there is a hypersphere $p \in P \cap C(P)$ and by the property M1 it follows $p \perp p$, i.e. p is a point, which is in contradiction with the hypothesis.

Theorem 30. If P is a regular subspace of the space $(S, \perp)$ and P_0 is a subspace of $(S, \perp)$ such that $P_0 \subseteq P$, then $\dim_{\perp} [P \cap C(P_0)] = \dim_{\perp} P - \dim_{\perp} P_0 - 2$.

PROOF. By Definition 14 $P \cap C(P) = \emptyset$ and hence $P_0 \cap C(P) = \emptyset$. Since $\dim_{\perp} \emptyset = -2$, the statement follows immediately from Theorem 28.

Theorem 31. If P is a regular subspace of the space $(S, \perp)$ with $\dim_{\perp} P = m$, $m \in \{0, 1, \dots, n-1\}$, and $\{p_1, \dots, p_k\}$, $k \in \{1, \dots, m+2\}$ is an $\perp$ -independent subset of P , then there is an $(m+1)$ -simplexoid with elements from P , so that $p_1, \dots, p_k$ are the elements of that $(m+1)$ -simplexoid.

PROOF. If $k < m+2$ then by Theorem 26 there are hyperspheres $p_{k+1}, \dots, p_{m+2} \in P$ such that $\{p_1, \dots, p_{m+2}\}$ is a basic set of the subspace P . For $k = m+2$ this statement is trivially realized. Let be now $P_{\alpha} = \langle p_1, \dots, p_{\alpha-1}, p_{\alpha+1}, \dots, p_{m+2} \rangle$ for $\forall \alpha \in \{1, \dots, m+2\}$. Then $P_{\alpha} \subset P$ and $\dim_{\perp} P_{\alpha} = m-1$. Hence by Theorem 30 $\dim_{\perp} [P \cap C(P_{\alpha})] = -1$, and by Theorem 23 it follows that the subspace $P \cap C(P_{\alpha})$ contains exactly one hypersphere. Denote this hypersphere with q_{α} . For $\forall \alpha \in \{1, \dots, m+2\}$ we have $q_{\alpha} \in P \cap C(P_{\alpha})$ and hence $q_{\alpha} \in C(P_{\alpha})$. By the property M1 it follows $\{p_1, \dots, p_{\alpha-1}, p_{\alpha+1}, \dots, p_{m+2}\} \perp q_{\alpha}$. Suppose now that $p_{\alpha} \perp q_{\alpha}$. Then $\{p_1, \dots, p_{m+2}\} \perp q_{\alpha}$. Since $\{p_1, \dots, p_{m+2}\}$ is the basic set of the subspace P , $q_{\alpha} \in C(P)$. From $q_{\alpha} \in P$ and $q_{\alpha} \in C(P)$ by Definition 14 we get that P is a singular subspace, contrary to hypothesis. Therefore

$$(\forall \alpha, \beta \in \{1, \dots, m+2\}) [p_{\alpha} \perp q_{\beta} \Leftrightarrow \alpha \neq \beta],$$

i.e. $(p_1, \dots, p_{m+2})$ is an $(m+1)$ -simplexoid with the elements from P .

Definition 15. If P is a subset of S and “ $\top$ ” a binary relation on P such that

$$(\forall p, q \in P) [p \top q \Leftrightarrow p \perp q],$$

then we say that “ $\top$ ” is the relation on P induced by the relation of orthogonality.

From Theorem 31 and Definitions 1, 2 and 15 it follows immediately

Theorem 32. If P is a regular subspace of the space $(S, \perp)$ and “ $\top$ ” is the relation on P induced by the relation of orthogonality, then every $\perp$ -independent subset of P is a $\top$ -independent set.

The converse of Theorem 32 is also true, even under weaker conditions.

Theorem 33. If P is a subspace of the space $(S, \perp)$ and " $\top$ " the relation on P induced by the relation of orthogonality, then every $\top$ -independent subset of P is an $\perp$ -independent set.

PROOF. Let $\{p_1, \dots, p_k\}$, $k \in \{1, \dots, m+2\}$ be a $\top$ -independent subset of P . Then by Definition 2 there is an $(m+1)$ -simplexoid $\mathfrak{P} = (p_1, \dots, p_{m+2})$ with elements $p_1, \dots, p_k$ and moreover some elements $p_{k+1}, \dots, p_{m+2} \in P$. Let $\mathfrak{Q} = d(\mathfrak{P})$, $\mathfrak{Q} = (q_1, \dots, q_{m+2})$. The statement of the theorem will be proved if we prove that $P_0 = \{p_1, \dots, p_{m+2}\}$ is an $\perp$ -independent set. Suppose, to the contrary, that P_0 is not an $\perp$ -independent set. By Theorem 19 there is an $\perp$ -independent subset $\bar{P}_0$ of P_0 such that the sets P_0 and $\bar{P}_0$ generate the same subspace Q of the space $(S, \perp)$. By symmetry we can suppose that $\bar{P}_0 = \{p_2, \dots, p_{m+2}\}$, $j \in \{2, \dots, m+2\}$. Since by Definition 1 $P_0 \perp q_1$ and hence $\bar{P}_0 \perp q_1$, and since $\bar{P}_0$ is a basic set of the subspace Q of $(S, \perp)$, hence $q_1 \in C(Q)$. Since on the other hand $p_1 \in P_0 \subseteq Q$, by the property M1 it follows $p_1 \perp q_1$, i.e. $p_1 \top q_1$, which contradicts Definition 5. This proves our statement.

Now we can prove

Theorem 34. If P is a regular subspace of the space $(S, \perp)$ with $\dim_{\perp} P = m$, $m \in \{0, 1, \dots, n-1\}$, and if " $\top$ " is the relation on P induced by the relation of orthogonality, then $(P, \top)$ is an m -dimensional Möbius space.

PROOF. The property S1 of the structure $(P, \top)$ follows from the same property of the structure $(S, \perp)$. Let $P_0 = \{p_1, \dots, p_{m+1}\}$ be any subset of P . Let further $Q = \langle P_0 \rangle$. Then $Q \subset P$ and $\dim_{\perp} Q \leq m-1$. Thus by Theorem 30 it follows $\dim_{\perp} [P \cap C(Q)] \leq -1$. This by Theorem 23 means that the subspace $P \cap C(Q)$ contains at least one hypersphere q . Then $q \in P$ and $q \in C(Q)$, and by the property M1 it follows $P_0 \perp q$, i.e. $P_0 \top q$. In particular $\{p_1, \dots, p_{m+1}\} \top q$, that proves the property S2 of the structure $(P, \top)$. If P is a $\top$ -independent set, then by Theorem 33 P_0 is an $\perp$ -independent set and hence $\dim_{\perp} Q = m-1$. Therefore $\dim_{\perp} [P \cap C(Q)] = -1$, and by Theorem 23 there is exactly one hypersphere $q \in P \cap C(Q)$, i.e. a hypersphere q such that $\{p_1, \dots, p_{m+1}\} \top q$, which proves the property S4 of the structure $(P, \top)$. Let now $\{p_1, \dots, p_{m+2}\}$ be any basic set of the subspace P . If there were a hypersphere $q \in P$ such that $\{p_1, \dots, p_{m+2}\} \top q$, i.e. $\{p_1, \dots, p_{m+2}\} \perp q$, it would be $q \in C(P)$, and hence $q \in P \cap C(P)$, which is impossible, while P is a regular subspace of the space $(S, \perp)$. Thus the property S3 of the structure $(P, \top)$ is proved, and by Definition 4 it follows that $(P, \top)$ is an m -dimensional Möbius space.

By the definition of the relation " $\top$ " and the definition of subspaces it follows immediately that every subspace of the space $(P, \top)$ is also a subspace of the space $(S, \perp)$ and conversely that every subspace of the space $(S, \perp)$ which is a subset of P , is also a subspace of the space $(P, \top)$.

Finally, by Theorems 32 and 33 we get

Theorem 35. If P is a regular subspace of the space $(S, \perp)$, " $\top$ " the relation on P induced by the relation of orthogonality, and Q any subspace of the space $(P, \top)$, and hence also a subspace of the space $(S, \perp)$, then $\dim_{\perp} Q = \dim_{\top} Q$.

Therefore, the structure of the space $(P, \top)$ is identical with the structure of the subspace P induced by the structure of the space $(S, \perp)$.

8. Remark

By the comparison of system of axioms S1—S4 and system of axioms P1—P4 from [7] it follows that an n -dimensional Möbius space is in fact an $(n+1)$ -dimensional pre-projective space. (About the pre-projective plane see the articles [1], [2] and [3] of V. DEVIDÉ.)

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