

The absolute annihilator of an abelian group modulo a subgroup

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A group G in this paper will always be meant to be an abelian group with addition to the group operation. H will denote a subgroup of G .

The absolute annihilator, $A(G)$, of a group G has been defined [2, problem 94] as follows: $A(G) = \{g \in G / Rg = gR = 0 \text{ for every ring } R \text{ on } G\}$. We consider here the following generalization of $A(G)$:

(i) Definition: The absolute annihilator of G modulo H , $(G:H) = \{g \in G / Rg \subseteq H, \text{ and } gR \subseteq H \text{ for every ring } R \text{ on } G\}$.

The absolute annihilator of G is clearly $(G:0)$.

In (ii) we will define the notion of the right and the left annihilator of G in the obvious manner. It will be shown that both these subgroups of G coincide with $(G:H)$.

A study of $(G:H)$ will be made for the case where H is an ideal in every ring on G in (iii). It will be shown that in this circumstance, the nilstufe [1, 5] of G and H are related.

$(G:H)$ will be employed to generalize the notion of a nil group (iv).

(ii) Definition: The right absolute annihilator of G modulo H , $(G:H)_r = \{g \in G / Rg \subseteq H \text{ for all rings } R \text{ on } H\}$. In a similar manner we define $(G:H)_l$ the left absolute annihilator of G modulo H .

Theorem 1. $(G:H) = (G:H)_r = (G:H)_l$.

PROOF. Since $(G:H) = (G:H)_r \cap (G:H)_l$ it suffices to show that $(G:H)_r = (G:H)_l$. Let $g \in (G:H)_r$ and let R be a ring on G , $R = (G, X)$, X a multiplication on G . For $a, b \in G$, define $a\bar{X}b = bXa$. $\bar{X}$ is a multiplication on G . Put $\bar{X} = (G, \bar{X})$. Since $g \in (G:H)_r$, $gXa = a\bar{X}g \in H$ for every $a \in G$. Therefore, $g \in (G:H)_l$ and hence $(G:H)_r \subseteq (G:H)_l$. Similarly, $(G:H)_l \subseteq (G:H)_r$.

(iii). Notation: Let $R_i = (G, X_i)$, X_i a multiplication on G_i , $1 \leq i \leq n$. Put $R_1(R_2(R_3(\dots R_n))) = \{g_1 X_1(g_2 X_2(g_3 \dots X_{n-1} g_n)) \dots \mid g_i \in G, 1 \leq i \leq n\}$, and $\underbrace{[G, \dots, G]}_{n\text{-times}} = \underbrace{(G : (G : (\dots (G:0)) \dots))}_{n\text{-times}}$.

We make the following obvious:

Observation 1: $R_1(R_2(R_3(\dots R_n))) = 0$ for all n rings R_i , $1 \leq i \leq n$, on G iff $\underbrace{[G, \dots, G]}_{n\text{-times}} = G$.

The nilstufe $v(G)$ of G is defined, [1, 5], to be n if there exists an associative ring R on G with $R^n \neq 0$, but $R^{n+1} = 0$ for every associative ring R on G . An obvious consequence of observation 1 is

Observation 2: If $\underbrace{[G, \dots, G]}_{n-1 \text{ times}} = 0$, then $v(G) \leq n$.

Clearly, H is an ideal in every ring on G iff $(G:H) \supseteq H$. In this case we obtain an ascending chain of subgroups of G , $H \subseteq (G:H) \subseteq (G:(G:H)) \subseteq (G:(G:(G:H))) \subseteq \dots$

If this chain terminates at G after a finite number of steps, then we have the following:

Theorem 2. If $(G:H) \supseteq H$ and if $\underbrace{(G:(G:(\dots(G:H))\dots))}_{n \text{ times}} = G$, then $v(G) < (n+1)(v(H)+1)$.

PROOF. Clearly $(G:(G:(\dots(G:H))\dots)) = G$ implies that $R^{n+1} \subseteq H$ for every ring R on G . Since H is an ideal in every ring on H , the restriction of every multiplication on G to H is a multiplication on H . Therefore $R^{(n+1)(v(H)+1)} \subseteq H^{v(H)+1} = 0$.

Example: Let $G = H \oplus K$, H a nil torsion group, K a torsion free group. It is easy to show that $\underbrace{(G:(G:(\dots(G:H))\dots))}_{v(K) \text{ times}} = G$.

Clearly, $(G:H) \supseteq H$. Therefore, by theorem 2, $v(G) < 2(v(K)+1)$. This result was obtained in [1, theorem 1].

We wish to show that the example just considered is the most general example of a mixed group that can be considered if we require that $v(G) < \infty$.

Theorem 3. Let G be a mixed group. If $v(G) < \infty$ then G splits, and the torsion part of G is nil and coincides with the maximal divisible subgroup of G .

PROOF. Let $G = D \oplus K$, D the maximal divisible subgroup of G . If D is not a torsion group then $v(D) = \infty$ and hence $v(G) = \infty$. Therefore D is a torsion group and hence nil [2, 4]. If K is not torsion free then K has a finite cyclic direct summand [3, theorem 9], in which case $v(K) = \infty$, and hence $v(G) = \infty$. Therefore K is torsion free, and D is the torsion part of G .

Theorem 4. Let $G = D \oplus K$, D the maximal divisible subgroup of G , and D nil. If $(G:K) \supseteq K$, then $v(G) = v(K)$.

PROOF. The following are easily shown to be true for every multiplication on G :

- a) $D^2 = 0$
- b) $DK \subseteq D$
- c) $KD \subseteq D$
- d) $K^{v(K)+1} \subseteq D$.

However since $(G:K) \supseteq K$ we have that $DK \subseteq K$ and $KD \subseteq K$. This together with b) and c) yields:

- b') $DK = 0$, and
- c') $KD = 0$.

$(G:K) \supseteq K$ further implies that $K^n \subseteq K$ for every positive integer n , which together with d) yields

$$d') \quad K^{v(K)+1} = 0$$

Let $g_i \in G$, $1 \leq i \leq v(K)+1$. $\prod_{i=1}^{v(K)+1} g_i$ is a finite sum of monomials

$$\prod_{i=1}^{v(K)+1} y_i, y \in D \text{ or } y_i \in K, 1 \leq i \leq v(K)+1.$$

b'), c'), and d') insure that

$$\prod_{i=1}^{v(K)+1} y_i = 0.$$

(iv) Definition: G is said to be a nil group modulo H if $(G:H) = G$.

A nil group in the ordinary sense is a nil group modulo 0.

Example: Let $G = H \oplus K$, H a nil torsion free group, K a torsion group. It is easy to show that G is nil modulo K .

Theorem 5. *Let G be nil modulo H . If H is divisible then G/H is nil.*

PROOF. Let $\Theta: G \rightarrow G/H$ be the canonical homomorphism. The sequence $0 \rightarrow H \rightarrow G \xrightarrow{\Theta} G/H \rightarrow 0$ is exact, hence the sequence

$$\text{Hom}(G \otimes G, G) \xrightarrow{\Theta^*} \text{Hom}(G \otimes G, G/H) \rightarrow \text{Ext}(G \otimes G, H) = 0$$

is exact. The last equality holds true since H is divisible. $\Theta^*(\mu) = \Theta\mu$ for all $\mu \in \text{Hom}(G \otimes G, G)$. We can identify an arbitrary multiplication on G/H with an element $\bar{\mu} \in \text{Hom}(G/H \otimes G/H, G/H)$ [2, theorem 118.1]. Let $g_1, g_2 \in G$, and let $\bar{g}_1, \bar{g}_2$ be their respective cosets in G/H .

$$\bar{\mu}(\bar{g}_1 \otimes \bar{g}_2) = \bar{\mu} \cdot (\Theta \otimes \Theta)(g_1 \otimes g_2).$$

$\bar{\mu} \cdot (\Theta \otimes \Theta) \in \text{Hom}(G \otimes G, G/H)$, and Θ^* is an epimorphism. Therefore there exists a $\mu \in \text{Hom}(G \otimes G, G)$ such that $\bar{\mu} \cdot (\Theta \otimes \Theta) = \Theta^*(\mu) = \Theta \cdot \mu$. Therefore, $\bar{\mu}(\bar{g}_1 \otimes \bar{g}_2) = \Theta\mu(g_1 \otimes g_2)$. G is nil modulo H , which implies that $\mu(g_1 \otimes g_2) \in H$, and hence $\bar{\mu}(\bar{g}_1 \otimes \bar{g}_2) = \Theta(\mu(g_1 \otimes g_2)) = 0$, or G/H is nil.

Theorem 6. *If $(G:H) \supseteq H$ and if G/H is a nil group, then G is nil modulo H .*

PROOF. Since $(G:H) \supseteq H$, H is an ideal in every ring on G , and hence every multiplication X_G on G induces a multiplication $X_{G/H}$ on G/H in a natural way. Let $g_1, g_2 \in G$, and let $\bar{g}_1, \bar{g}_2$ be their respective cosets in G/H . $\bar{g}_1 X_{G/H} \bar{g}_2 = \bar{g}_1 X_G g_2$. However G/H is a nil group, so that $\bar{g}_1 X_{G/H} \bar{g}_2 = 0$, which implies that $g_1 X_G g_2 \in H$, so that G is nil modulo H .

(v) The results obtained here suggest the following problems:

Problem 1: Characterize $(G:H)$ for H a special type of subgroup of G . For example, if G is a torsion group, and if G_p is the p -primary component of G , p a prime, then

$$(G:G_p) = G_p \otimes \sum_{\substack{q \text{ prime} \\ q \neq p}} G_q^1, \text{ where } G_q^1 \text{ denotes the first Ulm subgroup of } G_q.$$

Problem 2: Let K be a reduced torsion free group. Let $NS(K) = \{v(G)/G = D \oplus K, D \text{ a nil torsion group}\}$. By the example following theorem 2 we have that if $n \in NS(K)$ then $v(K) \leq n < 2(v(K) + 1)$. Is the converse true?

Problem 3: Find conditions for G to be nil modulo H , H a special type of subgroup of G .

Problem 4: If G is nil modulo H , must G/H be a nil group?

References

- [1] S. FEIGELSTOCK, On the Nilstufe of the Direct Sum of Two Groups, *Acta Math. Hung.*, **22** (1972), 449—452.
- [2] L. FUCHS, Infinite Abelian Groups, *New York and London*, vol. 1 (1970), vol. 2 (1973).
- [3] I. KAPLANSKY, Infinite Abelian Groups, *Ann Arbor*, (1954).
- [4] T. SZELE, Zur Theorie der Zeroringe, *Math. Ann.*, **121** (1949), 242—246.
- [5] T. SZELE, Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen, *Math. Zeit.*, **54** (1951), 168—180.

(Received May 13, 1974.)

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