

T-ideals based on multilinear identities

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This note shows that the T -ideal based on $f(x_1, \dots, x_n)$ is also based on multilinear elements when the characteristic of the field is zero or greater than $\max_{i=1, \dots, n} \{x_i \text{ degree of } f\}$. Furthermore this is essentially best possible.

Let K be a field. The free associative K -algebra with indeterminates $\{x_i\}_{i=1}^\infty$ and without constant term will be denoted by $K[X]$. The set of monomials of $K[X]$ forms a basis for $K[X]$ as a vector space over K and hence gives a canonical representation of the polynomials in $K[X]$. Define the x_i degree of a monomial to be the number of times x_i occurs in that monomial, and define the x_i degree (respectively, order) of f in $K[X]$ to be $\max \{x_i \text{ degree of } m : m \text{ is a monomial of } f\}$ (respectively, min). The degree of a monomial m is $\sum_i (x_i \text{ degree of } m)$. The degree (respectively order) of a polynomial f is $\max \{\text{degree of } m : m \text{ is a monomial of } f\}$ (respectively min). A polynomial f is linear in x_i if x_i occurs exactly once in each monomial, and f is multilinear if it is linear in each of its variables.

A T -ideal of $K[X]$ is any ideal which is mapped into itself by the endomorphisms of $K[X]$. If S is a subset of $K[X]$ then $T\langle S \rangle$ is the smallest T -ideal containing S . The elements of $T\langle S \rangle$ are exactly

$$\sum_i \alpha_i S_i(f_{i1}, \dots, f_{i n(i)}) \beta_i \text{ where } S_i \in S,$$

$f_{ij} \in K[X]$ and $\alpha_i, \beta_i \in K[X] \cup K$. If I is a T -ideal and $I = T\langle S \rangle$ then I is said to be based on S . An important tool (e.g. [2]) in the study of rings satisfying a polynomial identity f , is the fact that if the characteristic of K is zero, then $T\langle f \rangle = T\langle M \rangle$ where M is a subset of the multilinear elements of $T\langle f \rangle$. The first theorem shows that the characteristic of K need only be greater than $\max_i \{x_i \text{ degree of } f\}$.

Theorem 1. *Let $f \in K[X]$. If the characteristic of K is zero or greater than $\max_i \{x_i \text{ degree of } f\}$ then $T\langle f \rangle = T\langle M \rangle$ where M is a finite set of multilinear elements in $T\langle f \rangle$.*

The proof will follow from a lemma.

Lemma 2. *Let $f(x_1, \dots, x_n) \in K[X]$, and write $f = f_0 = h_0 + g_0$ where each monomial of h_0 has x_1 degree d with d the x_1 degree of f , and g_0 has x_1 degree less than d . Then*

there exists $h_{d-1}(x_1, y_1, \dots, y_{d-1}, x_2, \dots, x_n)$ contained in $T\langle f \rangle$, linear in $x_1, y_1, \dots, y_{d-1}$ such that $h_{d-1}(x_1, x_1, \dots, x_1, x_2, \dots, x_n) = d! h_0(x_1, x_2, \dots, x_n)$.

PROOF. The proof constructs h_{d-1} by the well known [1] linearization process. Suppose $f_i = h_i + g_i$ has been defined with $f_i \in T\langle f \rangle$, each monomial of $h_i = h_i(x_1, y_1, \dots, \dots, y_i, x_2, \dots, x_n)$ has x_1 degree $d-i$ and is linear in $y_1, \dots, y_i$, the x_i degree of g_i is less than $d-i$ and $h_i(x_1, x_1, \dots, x_1, x_2, \dots, x_n) = \frac{d!}{(d-i)!} h_0(x_1, x_2, \dots, x_n)$. Define $f_{i+1}(x_1, y_1, \dots, y_{i+1}, x_2, \dots, x_n) = f_i(x_1 + y_{i+1}, y_1, \dots, y_i, x_2, \dots, x_n) - f_i(x_1, y_1, \dots, \dots, y_i, x_2, \dots, x_n) - f_i(y_{i+1}, y_1, \dots, y_i, x_2, \dots, x_n)$. Thus $f_{i+1} \in T\langle f \rangle$ and $f_{i+1} = h_{i+1} + g_{i+1}$ where h_{i+1} is linear in $y_1, \dots, y_{i+1}$, each monomial of h_{i+1} has x_1 degree $d-(i+1)$ and g_{i+1} has x_1 degree less than $d-(i+1)$. Furthermore the monomials of h_{i+1} are obtained from monomials of h_i by substituting exactly one y_{i+1} in one of the positions occupied by an x_1 in the monomials of h_i . Since there are $d-i$ positions in which to substitute y_{i+1} , in each monomial, $h_{i+1}(x_1, y_1, \dots, \dots, y_i, x_1, x_2, \dots, x_n) = (d-i)h_i(x_1, y_1, \dots, y_i, x_2, \dots, x_n)$. Thus

$$\begin{aligned} h_{i+1}(x_1, x_1, \dots, x_1, x_2, \dots, x_n) &= (d-i)h_i(x_1, x_1, \dots, x_1, x_2, \dots, x_n) = \\ &= (d-i) \frac{d!}{(d-i)!} h_0(x_1, x_2, \dots, x_n) = \frac{d!}{(d-(i+1))!} h_0(x_1, \dots, x_n). \end{aligned}$$

Hence finally $f_{d-1} = h_{d-1} + g_{d-1}$ with $f_{d-1} \in T\langle f \rangle$ and g_{d-1} of degree 0 in x_1 . Thus by substituting 0 for x_1 we have $g_{d-1} \in T\langle f \rangle$ and $h_{d-1} \in T\langle f \rangle$ with $h_{d-1}(x_1, y_1, \dots, y_{d-1}, x_2, \dots, x_n)$ linear in $x_1, y_1, \dots, y_{d-1}$ and $h_{d-1}(x_1, x_1, \dots, x_1, x_2, \dots, x_n) = d! h_0(x_1, \dots, x_n)$.

PROOF OF THEOREM 1. As in the lemma $f = h_0 + g_0$ with h_0 of degree and order d in x_1 and g_0 of degree less than d . Thus $h_{d-1} \in T\langle f \rangle$ and since $d!$ is invertible in K , $h_0 \in T\langle h_{d-1} \rangle$, which in turn means $g_0 \in T\langle f \rangle$. By induction $T\langle g_0 \rangle = T\langle M_1 \rangle$ where M_1 is a finite set of multilinear elements in $T\langle g_0 \rangle \subseteq T\langle f \rangle$, and $T\langle h_{d-1} \rangle = T\langle M_2 \rangle$ where M_2 is a finite set of multilinear elements in $T\langle h_{d-1} \rangle \subseteq T\langle f \rangle$. Thus $T\langle f \rangle = T\langle h_{d-1}, g_0 \rangle = T\langle M_1 \cup M_2 \rangle$ and the theorem is proven.

To indicate that this is best possible we prove:

Theorem 3. $T\langle x^n \rangle = T\langle M \rangle$ where M is the set of multilinear identities in $T\langle x^n \rangle$ if and only if the characteristic of K is zero or greater than n .

Lemma 4. Let V_n be the set of multilinear identities of $T\langle x^n \rangle$ in the variables $x_1, \dots, x_n$. Then

$$V_n = \left\{ c \cdot \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)} : c \in K \right\}.$$

PROOF. It is easy to see that linearization process of lemma 2 yields $\sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)} \in V_n$. If $m \in V_n$ then $m = \sum_i \alpha_i f_i^n \beta_i$. Let $f_i = a_{i1}x_1 + \dots + a_{in}x_n + \gamma_i$ where the order of γ_i is greater than one, $a_{ij} \in K$. Thus

$$m = \sum_i c_i (a_{i1}x_1 + \dots + a_{in}x_n)^n + r$$

where $c_i, a_{ij} \in K$ and the order of r is greater than n . Since m has degree n , $r=0$. The multilinear monomials of $(a_{i1}x_1 + \dots + a_{in}x_n)^n$ are thus all possible $x_{\sigma(1)} \dots x_{\sigma(n)}$ $\sigma \in S_n$ with coefficient $a_{i1}a_{i2} \dots a_{in}$. Hence the multilinear part of $\sum_i c_i(a_{i1}x_1 + \dots + a_{in}x_n)^n$ is $\sum_i c_i(a_{i1} \dots a_{in}) \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)}$ which equals m . Thus

$$V_n = \left\{ c \cdot \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)} : c \in K \right\}.$$

PROOF OF THEOREM 3. By theorem 2, if the characteristic of K is zero or greater than n then $T\langle x^n \rangle = T\langle M \rangle$. Suppose now that the characteristic of K is k with $0 < k \leq n$. Any element q in $T\langle M \rangle$ equals $\sum_{\sigma \in S_n} f_{\sigma(1)} \dots f_{\sigma(n)} + g$ where f_i have degree one and g has order $n+1$. Thus to have monomials of degree n and x degree n , $f_i = c_i x + f'_i$, so that $q = c_1 \dots c_n \sum_{\sigma \in S_n} x^n + g' = n! c_1 \dots c_n x^n + g'$ where g' has no monomial of degree n which is also of x degree n . Thus for x^n to be in $T\langle M \rangle$ requires $g' = 0$ and $n! c_1 \dots c_n = 1$. Since $0 < k \leq n$, this last condition is never satisfied.

References

- [1] N. JACOBSON, Structure of Rings, *Amer. Math. Soc. Colloquium Publ.* 37, 1964.
- [2] D. KRAKOWSKI and A. REGEV, The polynomial identities of a Grassman algebra. *Trans. Amer. Math. Soc.* 182, (1973).

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