

On the mean convergence of interpolatory processes

By P. VÉRTESI (Budapest)

1. Introduction and preliminary results.

1.1. Let us denote by

$$(1.1) \quad x_{kn} \equiv \cos \vartheta_{kn} = \cos \frac{2k-1}{2n} \pi \quad (k = 1, 2, \dots, n; n = 1, 2, \dots)$$

the roots of the Chebysheff polynomials $T_n(x) = \cos n\vartheta$ ($x = \cos \vartheta$), $L_n(f; x) = \sum_{k=1}^n f(x_{kn}) l_{kn}(x)$ the Lagrange interpolatory polynomials of degree $\leq n-1$, $H_n(f; x) = \sum_{k=1}^n f(x_{kn}) h_{kn}(x)$ the Hermite—Fejér interpolatory polynomials of degree $\leq 2n-1$. Here

$$(1.2) \quad l_{kn}(x) = \frac{T_n(x)}{T'_n(x_{kn})(x-x_{kn})} = \frac{(-1)^{k+1} T_n(x) \sqrt{1-x_{kn}^2}}{n(x-x_{kn})} \quad (k = 1, 2, \dots, n),$$

(1.3)

$$h_{kn}(x) = \left[1 - \frac{T''_n(x_{kn})}{T'_n(x_{kn})} (x-x_{kn}) \right] l_{kn}^2(x) = \left[\frac{T_n(x)}{n(x-x_{kn})} \right]^2 (1-xx_{kn}) \quad (k = 1, 2, \dots, n).$$

As it is well known

$$(1.4) \quad L_n(f; x_{kn}) = f(x_{kn}) \quad (k = 1, 2, \dots, n; n = 1, 2, \dots),$$

$$(1.5) \quad H_n(f; x_{kn}) = f(x_{kn}), \quad H'_n(f; x_{kn}) = 0 \quad (k = 1, 2, \dots, n; n = 1, 2, \dots).$$

1.2. According with a classical result of L. FEJÉR [1] if $f(x) \in C[-1, 1]$ (i.e. $f(x)$ is a continuous function on $[-1, 1]$) we have with the notation $\|g(x)\| = \max_{-1 \leq x \leq 1} |g(x)|$

$$(1.6) \quad \lim_{n \rightarrow \infty} \|H_n(f; x) - f(x)\| = 0.$$

So

$$(1.7) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 [H_n(f; x) - f(x)]^2 \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } f \in C[-1, 1].$$

For the L_n process as a special case of the paper P. Erdős and P. Turán shows ([2], (13))

$$(1.8) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 [L_n(f; x) - f(x)]^2 \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } f \in C[-1, 1].$$

1.3. In his paper [3] P. Turán investigated the quasi Hermite—Fejér process from the point of view of uniform convergence. More exactly he considered the polynomial $H_n^*(f; x)$ of degree $\leq 2n-2$ whose first derivative vanishes at all nodes except at the exceptional $\eta_{1(n)} = x_{r(n), n}$, i.e.

$$(1.9) \quad H_n^*(f; x_{kn}) = f(x_{kn}) \quad (k = 1, 2, \dots, n); \quad H_n^{*'}(f; x_{kn}) = 0 \quad (x_{kn} \neq \eta_{1(n)}).$$

Later in [4] we considered the polynomial $H_n^{**}(f; x)$ of degree $\leq 2n-3$ such that

$$(1.10) \quad H_n^{**}(f; x_{kn}) = f(x_{kn}) \quad (k = 1, 2, \dots, n); \quad H_n^{**'}(f; x_{kn}) = 0 \quad (x_{kn} \neq \eta_{1(n)}, \eta_{2(n)}).$$

In both papers phenomena were found which were unexpected after (1.6). (In the papers [5] and [6] were considered the processes whose first derivative vanishes at all nodes except k or $m_n(m_n \nearrow \infty)$ exceptional roots.)

1.4. In connection with the good mean convergence behaviour of the L_n and H_n processes (see. (1.7) and (1.8)) P. TURÁN asked whether the "middle" processes (H_n^* , H_n^{**} , a.s.o.) preserve or not the mentioned good convergence property. The aim of this paper to give some theorems in this direction.

2. New results.

The following two theorems show that the H_n^* and the H_n^{**} sometimes are not good processes.

Theorem 2.1. *If p is a fix positive even integer then for any $f \in C[-1, 1]$*

$$(2.1) \quad \left\{ \int_{-1}^1 [H_n^*(f; x) - H_n(f; x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} = \begin{cases} O(n^{-\frac{1}{p}}) & \text{if } |\eta_{1(n)}| \leq 1 - \varepsilon \quad (\varepsilon > 0), \\ O(n^{1-\frac{1}{p}}) & \text{if } \eta_{1(n)} = x_{1n}, \end{cases}$$

or for the convergence of H_n^* process we have

(2.2)

$$\left\{ \int_{-1}^1 [H_n^*(f; x) - f(x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} = \begin{cases} o(1) + O(n^{-\frac{1}{p}}) & \text{if } |\eta_{1(n)}| \leq 1 - \varepsilon \quad (\varepsilon > 0), \\ O(n^{1-\frac{1}{p}}) & \text{if } \eta_{1(n)} = x_{1n}. \end{cases}$$

Further for a suitable $f_1(x) \in C[-1, 1]$ we get¹⁾

$$(2.3) \quad \left\{ \int_{-1}^1 [H_n^*(f_1; x) - H_n(f_1; x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} \sim \begin{cases} n^{-\frac{1}{p}} & \text{if } |\eta_{1(n)}| \leq 1 - \varepsilon \quad (\varepsilon > 0), \\ n^{1-\frac{1}{p}} & \text{if } \eta_{1(n)} = x_{1n} \end{cases}$$

¹⁾ $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$.

Theorem 2.2. *If p is a fix positive even integer then we can choose the sequences $\{\eta_{1(n)}\}$ and $\{\eta_{2(n)}\}$ such that for any $f \in C[-1, 1]$*

(2.4)

$$\left\{ \int_{-1}^1 [H_n^{**}(f; x) - H_n(f; x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} = \begin{cases} O(n^{-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ O(n^{1-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ O(n^{3-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \end{cases}$$

or

(2.5)

$$\left\{ \int_{-1}^1 [H_n^{**}(f; x) - f(x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} = \begin{cases} o(1) + O(n^{-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ O(n^{1-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ O(n^{3-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}. \end{cases}$$

Further for a suitable $f_2 \in C[-1, 1]$

(2.6)

$$\left\{ \int_{-1}^1 [H_n^{**}(f_2; x) - H_n(f_2; x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} \sim \begin{cases} n^{-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ n^{1-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}, \\ n^{3-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\}, \{\eta_{2(n)}\}. \end{cases}$$

Finally we mention the following interesting result.

Theorem 2.3. *For any $f \in C[-1, 1]$*

$$(2.7) \quad \int_{-1}^1 [H_n^*(f; x) - H_n(f; x)] \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^1 [H_n^{**}(f; x) - H_n(f; x)] \frac{dx}{\sqrt{1-x^2}} = 0$$

or

$$(2.8) \quad \int_{-1}^1 [H_n^*(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^1 [H_n^{**}(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} = \\ = \int_{-1}^1 [H_n(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} = o(1)$$

further

$$(2.9) \quad \left| \int_{-1}^1 [L_n(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} \right| = O\left(\omega_m\left(f; \frac{1}{n}\right)\right)$$

or

$$(2.10) \quad \int_{-1}^1 [L_n(f; x) - H_n(f; x)] \frac{dx}{\sqrt{1-x^2}} = o(1).$$

Here $\omega_m(f; t)$ is the m -th modulus of smoothness of $f(x)$.

Remark. Some theorems at $p = \infty$ were proved for the H_n^* and the H_n^{**} in [3] and [4].

3. PROOFS.

3.1. PROOF of theorem 2.1. As Turán proved in [3], (5.1)

$$(3.1) \quad H_n^*(f; x) = \sum_{k=1}^n f(x_{kn}) h_{kn}(x) + \frac{T_n^2(x)}{x - \eta_1} \frac{1}{n^2} \sum_{k=1}^n x_{kn} f(x_{kn}),$$

so by $H_n(f; x) = \sum_{k=1}^n f(x_{kn}) h_{kn}(x)$ for any positive even integer p

(3.2)

$$\begin{aligned} \left\{ \int_{-1}^1 [H_n^*(f; x) - H_n(f; x)]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} &= \left\{ \int_{-1}^1 \left[\frac{T_n^2(x)}{x - \eta_1} \frac{1}{n^2} \sum_{k=1}^n x_{kn} f(x_{kn}) \right]^p \frac{dx}{\sqrt{1-x^2}} \right\}^{\frac{1}{p}} = \\ &= \frac{1}{n^2} \left| \sum_{k=1}^n x_{kn} f(x_{kn}) \right| \left[\int_{-1}^1 \frac{T_n^{2p}(x)}{(x - \eta_1)^p \sqrt{1-x^2}} dx \right]^{\frac{1}{p}} = W. \end{aligned}$$

Here $\sum_{k=1}^n x_{kn} f(x_{kn}) = O(n)$ further using for the polynomial $Q_{2np-p} = T_n^{2p}(x)(x - \eta_1)^{-p}$ of degree $2np - p$ the Gauss—Jacobi quadrature formula we have

$$W \cong \frac{c}{n} \left[\frac{\pi}{np} \sum_{k=1}^{np} \frac{T_n^{2p} \left(\cos \frac{2k-1}{2np} \pi \right)}{\left(\cos \frac{2k-1}{2np} \pi - \eta_1 \right)^p} \right]^{\frac{1}{p}} = V.$$

Now $\left| T_n \left(\cos \frac{2k-1}{2np} \pi \right) \right| \cong c_1 > 0$, $\cos \frac{2k-1}{2np} \pi - \cos \frac{2s-1}{2n} \pi = -2 \sin \left(\frac{2k-1}{4np} + \frac{2s-1}{4n} \right) \pi \sin \left(\frac{2k-1}{4np} - \frac{2s-1}{4n} \right) \pi$ where $\eta_1 = \cos \frac{2s-1}{2n} \pi$. If $|\eta_1| \cong 1 - \varepsilon$ then $s \sim n$ on the other hand for $\eta_1 = x_{1n}$, $s = 1$. So

$$V \sim \frac{1}{n} \left[\frac{1}{n} \sum_{\substack{k=1 \\ k \neq s}}^n \frac{1}{\left(\frac{k-s}{n} \right)^p \left(\frac{k+s}{n} \right)^p} \right]^{\frac{1}{p}} \sim \begin{cases} n^{-\frac{1}{p}} & \text{for } |\eta_1| \cong 1 - \varepsilon, \\ n^{1-\frac{1}{p}} & \text{for } \eta_1 = x_{1n} \end{cases}$$

according with (2.1).

To prove (2.2) let us consider that

$$(3.3) \quad H_n^*(f; x) - f(x) = H_n^*(f; x) - H_n(f; x) H_n(f; x) - f(x).$$

Using (1.7) and (2.1) we obtain (2.2).

Finally, if we take the function $f_1(x) = x$ into account we get that $\sum_{k=1}^n x_{kn} f_1(x_{kn}) = \sum_{k=1}^n x_{kn}^2 \sim n$. The remaining parts are the same as above.

3.2. PROOF of theorem 2.2. As we proved in [4], (3.10)

$$\begin{aligned} H_n^{**}(f; x) = \\ = \sum_{k=1}^n f(x_{kn}) h_{kn}(x) + \frac{T_n^2(x)}{(x-\eta_1)(\eta_2-\eta_1)n^2} \left[\eta_2 \sum_{j=1}^n f(x_{jn}) x_{jn} + \sum_{j=1}^n f(x_{jn})(1-2x_{jn}^2) \right] + \\ + \frac{T_n^2(x)}{(x-\eta_2)(\eta_1-\eta_2)n^2} \left[\eta_1 \sum_{j=1}^n f(x_{jn}) x_{jn} + \sum_{j=1}^n f(x_{jn})(1-2x_{jn}^2) \right]. \end{aligned}$$

To prove the relations (2.4) and (2.5) let e.g. $\eta_1(n) = x\left[\frac{n}{4}\right]_n$, $\eta_2 = x\left[\frac{n}{2}\right]_n$ (so $\eta_1 - \eta_2 \cong \cong c > 0$); $\eta_1 = x\left[\frac{n}{4}\right]_n$, $\eta_2 = x\left[\frac{n}{4}\right]_{+1,n}$ (so $\eta_1 - \eta_2 \sim \frac{1}{n}$); $\eta_1 = x_{1n}$, $\eta_2 = x_{2n}$ (so $\eta_1 - \eta_2 \sim \frac{1}{n^2}$), respectively. Using that $[...] = O(n)$ and the above mentioned quadrature formula, we obtain the relations.

For (2.6) we can use the same η 's and the function $f_2(x) = 1 - 2x^2$ because $\sum_{k=1}^n x_k \cdot (1 - 2x_{kn}^2) = 0$ and $\sum_{k=1}^n (1 - 2x_{kn}^2)^2 \sim n$.

The remaining part is the same as above.

3.3. PROOF of theorem 2.3. By $Q_{n-1}(x) = \frac{T_n(x)}{(x-\eta_1)n^2} \sum_{k=1}^n x_{kn} f(x_{kn})$ we obtain from (3.1)

$$\int_{-1}^1 [H_n^*(f; x) - H_n(f; x)] \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^1 T_n(x) Q_{n-1}(x) \frac{dx}{\sqrt{1-x^2}} = 0.$$

We can use similar argument for H_n^{**} . We get (2.8) by (3.3) and (2.7). To prove (2.9) let $P_n(f; x)$ be a polynomial such that $\max_{-1 \leq x \leq 1} |f(x) - P_n(f; x)| \leq c \omega_m\left(f; \frac{1}{n}\right)$. Then

$$\begin{aligned} \left| \int_{-1}^1 [L_n(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} \right| &= \left| \int_{-1}^1 L_n(f; x) \frac{dx}{\sqrt{1-x^2}} - \int_{-1}^1 P_n(f; x) \frac{dx}{\sqrt{1-x^2}} + \right. \\ &+ \left. \int_{-1}^1 [P_n(f; x) - f(x)] \frac{dx}{\sqrt{1-x^2}} \right| \leq \left| \frac{\pi}{n} \sum_{k=1}^n f(x_{kn}) - \frac{\pi}{n} \sum_{k=1}^n P_n(f; x_{kn}) \right| + c_1 \omega_m\left(f; \frac{1}{n}\right) \leq \\ &\leq c_2 \omega_m\left(f; \frac{1}{n}\right) \end{aligned}$$

Using that $L_n(f; x) - H_n(f; x) = L_n(f; x) - f(x) + f(x) - H_n(f; x)$ and (1.6) we get (2.10).

References

- [1] L. FEJÉR, Die Abschätzung eines Polynomes, *Math. Z.*, **32** (1930), 426—457.
- [2] P. ERDŐS—P. TURÁN, On interpolation, *Ann. of Math.*, **38** (1937), 142—155.
- [3] P. TURÁN, A remark on Hermite—Fejér interpolation, *Ann. Univ. Budapest, (Sectio Math.)*, **3—4** (1960/61), 369—377.
- [4] P. VÉRTESI, On a problem of P. Turán, *Canad. Math. Bull.* **180** (1975), 283—288.
- [5] A. MEIR, A. SHARMA, J. TZIMBALARIO, Hermite—Fejér type interpolation processes, *Analysis Math.* (in press).
- [6] P. VÉRTESI, Hermite—Fejér interpolation omitting some derivatives, *Acta Math. Acad. Sci. Hungar.*, **26** (1975), 199—204.

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST. REÁLTANODA U. 13—15,
HUNGARY

(Received 22 May, 1974)