Homology of a differential algebra

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In this note we prove a theorem about the homology of a differential algebra over Z_2. The usefulness of this result lies in computing higher terms of spectral sequences; especially when the term from which we deserve the higher terms is a polynomial algebra. As a simple example, using the relation between homology operations and the dual steenrod squares, we obtain the E^4-term of the spectral sequences $K_*(\Omega^2 S^{k+2}; H_*(pt; Z_2)) \Rightarrow K_*(\Omega^2 S^{k+2}; Z_2)$ (this is the dual of the spectral sequence in [1]).

1. As in [2] a differential graded Algebra $A=(A, d)$ over K will mean a graded algebra A equipped with a K-module homomorphism d; $A \to A$ of degree -1 with $d^2=0$ and $d(uv)=duv+(-1)^{\deg u} udv$ holds. If A is such a differential algebra then $A[x]$ is made into a differential algebra with the differential d such that $dA \subset A$, $dx=y \in A$, $dy=0$ and y is not a zero divisor in $H(A)$. We shall denote by $\{y\}$ the homology class of y in $H(A)$.

Theorem 1.1. *Suppose that A is a differential algebra over Z_2 and $A[x)$ the corresponding differential algebra defined above. Then we have the isomorphism*

$$H(A[x]) \simeq \frac{H(A)}{(\{y\})}[x^2].$$

PROOF. We note that there is a short exact sequence of A-modules $0 \to A(x^2) \to \to A[x] \to xA[x^2] \to 0$ which gives rise to the exact triangle.

$$\begin{tikzcd} H(xA[x^2]) \arrow[rr, "\partial"] & & H(A[x^2]) \arrow[dl, "i"] \\ & H(A[x]) \arrow[ul, "j_*"] & \end{tikzcd}$$

where ∂ is the connecting map.

Simple calculation shows that $\partial\colon H(x \cdot A[x^2]) \to H(xA[x^2])$ is just multiplication by y/x and so the image of ∂ is $(\{y\})$. Because y is not a zero divisor in $H(A)$ this means that ∂ is one to one. Now, image of $j_* = \text{kernel of } \partial = \{0\}$ so that j_* is the zero map and $\ker j_* = \text{image } i_* = H(A[x])$ which implies that i_* is onto. Hence

$$\frac{H(A[x^2])}{(\{y\})} = \frac{H(A[x^2])}{\text{image } \partial} = \frac{H(A[x^2])}{\ker i_*} \simeq H(A[x]).$$

But since $H(A[x^2]) \simeq H(A)[x^2]$ we see that the theorem holds.

2. *Application.* Using homology operations $Q_i\colon H_n(X; Z_2) \to H_{2n+i}(X; Z_2)$ Browder (for details see [3]) showed that

$$H_*(\Omega^n S^{k+n}; Z_2) = Z_2[Q_i^{i_1} \dots Q_{n-1}^{i_n}(x_k)],\ x_k$$

is a generator of

$$H_*(S^k; Z_2).$$

If Sq_*^n denotes the dual steenrod square and $Q^j\colon H_n(X; Z_2) \to H_{n+j}(X; Z_2)$ is the homology operation such that $Q_x^j = Q_{j-n}x$, $x \in H_n(X; Z_2)$ then Nishida [4] proves that $Sq_*^n Q^{n+s} x = \sum \binom{s}{n-2i} Q^{s+t} Sq_*^i x$.

Using this relation it is not difficult to prove ([5]).

Lemma 2.1.

In the Atiyah—Hirzebruch sepectral sequence $H_*(\Omega^2 S^{k+2}; Z_2) \Rightarrow K_*(\Omega^2 S^{k+2}; Z_2)$ *we have* $d^3 Q_1^{p+2} x_k = (Q_1^p x_k)^4$, $Q_1^{p+2} = Q_1^{p+2} \dots Q_1$ and d^3 *is the third differential*

Since $H_*(\Omega^2 S^{k+2}; Z_2) = Z_2[x_k, Q_1 x_k, Q_1^2 x_k, \dots]$ we now obtain the E^4-term of the spectral sequence

Theorem 2.2. *The E^4-term of the Atiyah—Hirzebruch spectral sequence*

$$H_*(\Omega^2 S^{k+2}; Z_2) \Rightarrow K_*(\Omega^2 S^{k+2}; Z_2) \text{ is}$$

$$\frac{Z_2[x_k, Q_1 x_k, (Q_1^2 x_k)^2, \dots]}{(x_k^4, (Q_1 x_k)^4 \dots)} \quad \text{for } k \text{ odd},$$

$$\frac{Z_2[x_k, Q_1 x_k, (Q_1^2 x_k)^2, \dots]}{((Q_1 x_k)^4, (Q_1^2 x_k)^4, \dots)} \quad \text{for } k \text{ even}.$$

Proof. We prove this theorem for k odd since the proof is similar in the other case. If we let $A = Z_2[x_k, Q_1 x_k, Q_1^2 x_k, \dots]$ we can filter it by setting

$$A_1 = Z_2[x_k]$$

$$A_2 = Z_2[x_k, Q_1 x_k]$$

$$A_n = Z_2[x_k, Q_1 x_k, \dots Q_1^{n-1} x_k].$$

Since $d^3 Q_2^{p+1} x_k = (Q_1^p x_k)^4$ the theorem is trivially true for A_1 and A_2. Suppose that the theorem holds for A_n, that is

$$H(A_n) = \frac{Z_2[x_k, Q_1 x_k, (Q_1^2 x_k)^2, \dots, (Q^{n-1} x_k)^2]}{(x_k^4, (Q_1 x_k)^4, \dots (Q_1^{n-3} x_k)^4)}$$

We complete the proof by showing that the theorem also holds for A_{n+1}.
Note that $H(A_{n-1})=H(A_n[Q_1^n x_k])$.

Theorem 1.1 now implies that

$$H(A_n[Q_1^n x_k]) = \frac{H(A_n)}{(Q_1^{n-2}x_k)^4}[(Q_1^n x_k)^2] = \frac{Z_2[x_k, Q_1 x_k, (Q_1^2 x_k)^2, \dots (Q_1^n x_k)^2]}{((x_k)^4, (Q_1 x_k)^4, \dots, (Q_1^{n-2} x_k)^4)}.$$

References

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