

On a subsequence of primes

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To the memory of Prof. A. Kertész

1. Let P denote the sequence of primes. A subsequence $\{q_n\}$ of P satisfying $3 \leq q_1 < q_2 < \dots$ and

$$(1.1) \quad q_n \not\equiv 1 \pmod{q_i} \quad 1 \leq i < n, n \geq 2$$

will be called here a G -sequence. For a sequence $B = \{b_n\}$ we denote by $A(B, x)$ the number of elements of B not exceeding x .

In [1] S. W. GOLOMB studied the density of G -sequences and he proved that there does not exist a constant $A > 0$ such that

$$(1.2) \quad A(G, x) > Ax/\log x$$

for all sufficiently large x .

A special example is the sequence G_1 defined inductively by $q_1 = 3$ and q_n for $n \geq 2$ is the smallest prime greater than q_{n-1} for which $q_n \not\equiv 1 \pmod{q_i}$, $1 \leq i \leq n$. ERDŐS [2] proved for the sequence G_1

$$(1.3) \quad A(G_1, x) = (1 + o(1))x(\log x \cdot \log \log x)^{-1}.$$

H. G. MEIJER [3] sharpened the inequality (1.2). He proved that there does not exist a constant $A > 1$ such that

$$(1.4) \quad \frac{A(G, x) \cdot \log x \cdot \log \log x}{x} > A$$

for all sufficiently large x .

Furthermore he proved that there exists a G -sequence such that

$$(1.5) \quad \varliminf_{x \rightarrow \infty} \frac{A(G, x) \log x}{x} > c,$$

c being a positive constant.

Our aim is to prove the following

Theorem. For every G -sequence we have

$$(1.6) \quad \varlimsup_x \frac{A(G, x)}{x/\log x} < 1.$$

Furthermore, if ε is an arbitrary positive number, then there exists a G -sequence such that

$$(1.7) \quad \varlimsup_x \frac{A(G, x)}{x/\log x} > 1 - \varepsilon.$$

2. PROOF. The proof of (1.6) is very simple. Let $q_1 < q_2 < \dots < q_N$ be the first elements of G . Then $A(G, x)$ is not greater than the number of those primes p not exceeding x for which

$$p \not\equiv 1 \pmod{q_i} \quad (i = 1, \dots, N).$$

Using the erathostenian sieve and the prime number theorem for arithmetical progressions we get

$$A(G, x) \equiv (1 + o(1)) \frac{x}{\log x} \prod_{i=1}^N \frac{q_i - 2}{q_i - 1}.$$

For $x \rightarrow \infty$, we get (1.6).

Now we prove (1.7).

Let θ be a small positive number, $\theta < \frac{4}{\varepsilon}$. We shall construct a sequence of positive numbers $\{x_k\}$ tending to infinity,

$$x_1 < x_2 < \dots, \quad \theta x_k > x_{k-1}$$

and a G -sequence entirely contained in the union of the intervals $(\theta x_k, x_k)$ such that (1.7) holds for this G -sequence.

For $Y > 3/\theta$ let

$$(2.1) \quad L(y, \theta) = \prod_{\theta y < p < y} \left(1 - \frac{1}{p-1}\right).$$

It is well known that

$$(2.2) \quad (1 \equiv) L(y, \theta) \equiv 1 - c \frac{\log \theta}{\log y},$$

c being a positive constant.

Let $\varepsilon_1 \equiv \varepsilon_2 \equiv \dots$ be a sequence of positive numbers such that

$$(2.3) \quad \prod_{k=1}^{\infty} (1 - \varepsilon_k) > \left(1 - \frac{\varepsilon}{2}\right).$$

We shall choose the sequence x_k such that

$$(2.4) \quad L(x_k, \theta) \equiv 1 - \varepsilon_k \quad (k = 1, 2, \dots).$$

Suppose that we have already chosen $x_1, \dots, x_{k-1}$ ($k \equiv 1$) and the primes $q_1 < \dots < q_N$ of the sequence G contained in

$$\bigcup_{j=1}^{k-1} (\theta x_j, x_j).$$

Let T_k be the set of primes p satisfying

$$(2.5) \quad p \not\equiv 1 \pmod{q_i} \quad (i = 1, \dots, N).$$

The number of the primes p in the interval $\theta x < p < x$ satisfying (2.5) is asymptotically

$$(2.6) \quad (1 - \theta) \frac{x}{\log x} \prod_{i=1}^N \frac{q_i - 2}{q_i - 1}.$$

Let x be so large that the number of T_k in the interval $(\theta x, x)$ is greater than

$$\left(1 - \frac{3}{2}\theta\right) \frac{x}{\log x} \prod_{i=1}^N \frac{q_i - 2}{q_i - 1}.$$

Let $S_{k,x}$ be the set of those primes p' in $(\theta x, x)$ for which there exists at least one $p \in T_k$ such that

$$(2.7) \quad p' \equiv 1 \pmod{p}.$$

The number of $S_{k,x}$ does not exceed the number of solutions of the equation

$$(2.8) \quad p' - 1 = ap \quad 1 \leq a \leq 1/\theta, \quad p \leq x.$$

Using the Brun—Selberg sieve method (see [4]) we see that (2.8) has at most

$$(2.9) \quad c \sum_{a < 1/\theta} \frac{x}{\varphi(a) \log^2 x} = O\left(\frac{x}{\log^2 x} \log \frac{1}{\theta}\right)$$

of solutions.

If x is large enough then the right hand side of (2.9) is smaller than

$$(2.11) \quad \frac{\theta}{2} \frac{x}{\log x} \prod_{i=1}^N \frac{q_i - 2}{q_i - 1}.$$

Let now $x = x_k$ be so large that (2.5) and the other two conditions are satisfied. We continue the sequence G by all of the remained primes in $(\theta x_k, x_k)$. Hence

$$(2.12) \quad A(G, x_k) \equiv A(G, x_k) - A(G, \theta x_k) \equiv (1 - 2\theta) \frac{x}{\log x} \prod_{i=1}^N \frac{q_i - 2}{q_i - 1}.$$

Observing that

$$\prod_{i=1}^N \frac{q_i - 2}{q_i - 1} > \prod_{j=1}^{k-1} L(x_j, \theta) > 1 - \frac{\varepsilon}{2},$$

and $(1 - 2\theta) \equiv 1 - \frac{\varepsilon}{2}$, we get the inequality (1.7).

References

- [1] S. W. GOLOMB, Sets of primes with intermediate density, *Math. Scand* **3** (1955), 264—274.
- [2] P. ERDŐS, On a problem of Golomb, *J. Austral. Math. Soc.* **2** (1961/62), 1—8.
- [3] H. G. MEIJER, Sets of primes with intermediate density, *Math. Scand.* **34** (1974), 37—43.
- [4] K. PRACHAR, *Primzahlverteilung*, Berlin, 1957.

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