Equivalent topological properties of the space of signatures of a semilocal ring

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Dedicated to the memory of Andor Kertész

1. *Introduction and Notations*

Let R denote a (not necessarily noetherian) connected semilocal ring, $U(R)$ its group of units, and $X=X(R)$ the topological space of signatures of R [8, Def. 2.1]. The space X is compact, Hausdorff, and totally disconnected, with a subbase of the topology being given by the subsets $W(a)=\{\sigma \text{ in } X \mid \sigma(a)=-1\}$, a in $U(R)$, of X.

By a *space* over R we will mean a pair (E, B) where E is a projective (whence free) R-module and B is a nondegenerate symmetric bilinear form on E. Isometries of spaces will be written as $\cong$ and for any natural number m the space $E \perp E \perp \ldots \perp E$ (m times) will be denoted by mE or $m(E, B)$. An element e of E is called *primitive* if it can be augmented to a basis of E. The space (E, B) is called *isotropic* if it contains a primitive element e with $B(e, e)=0$. We will write $E=\langle a_1, \ldots, a_n\rangle$ to mean E has an orthogonal basis $e_1, \ldots, e_n$ such that $B(e_i, e_i)=a_i$ in $U(R)$. The Witt ring of equivalence classes of spaces will be denoted by $W(R)$ and the representative of a space (E, B) in $W(R)$ by $[E]$ or $[B]$. If E is a space then there exist $a_1, \ldots, a_n$ in $U(R)$ such that $[E]=[\langle a_1, \ldots, a_n\rangle]$ in $W(R)$ [7, Thm. 1. 16]. Any σ in $X(R)$ induces a ring homomorphism $\bar{\sigma}\colon W(R)\to \mathbf{Z}$ via $\bar{\sigma}([\langle a_1, \ldots, a_n\rangle])=\sigma(a_1)+\ldots+\sigma(a_n)$.

In [8, Thm. 3. 20], the following Strong Approximation Property was introduced:

SAP If Y is a clopen (closed and open) subset of X, then $Y=W(a)$ for some a in $U(R)$.

Following Elman and Lam [4, Def. 1.5], X is said to satisfy the Weak Approximation Property if

WAP The family of subsets $W(a)$ of X, as a runs through $U(R)$, forms a basis of the topology of X.

It is easy to see (cf. [8, Thm. 3. 20 and 4, p. 1159], that SAP is equivalent to: If Y_1, Y_2 are two disjoint closed subsets of X, there is an element a in $U(R)$ with $\sigma(a)=1$ for all σ in Y_1, and $\tau(a)=-1$ for all τ in Y_2. Moreover, WAP is equivalent to: For any closed subset Y of X and any τ in $X-Y$, there is an element a in $U(R)$ with $\sigma(a)=1$ for all σ in Y and $\tau(a)=-1$.

If R is a formally real field ELMAN and LAM [4, p. 1184] and PRESTEL [10, p. 319], independently, introduced the following Hasse—Minkowski Property.

HMP R satisfies HMP, if for every space (E, B) such that $|\bar{\sigma}([E])| \prec \operatorname{rank} E$, for all σ in X, there is a natural number m such that $m(E, B)$ is isotropic.

It is clear that SAP implies WAP and, in [4, Thm. 3.5], it was shown that if R is a formally real field SAP and WAP are equivalent. If, in addition, R is also pythagorean, [4, Thm. 5.3] shows that HMP is equivalent to the other two conditions, and, indeed, for arbitrary formally real fields, [10, Satz 3.1 and 5, Thm. C] demontrates the equivalence of all three conditions. The proofs depend on [3, Thms. 2.6 and 4.8] which seem to be quite deep, and, in particular, Theorem 4.8 of [3] relies on the Hauptsatz of ARASON and PFISTER.

The purpose of this note is to prove the equivalence of WAP, SAP, and HMP for connected semilocal rings with 2 in $U(R)$, using only basic facts about spaces and $W(R)$, thus resulting in considerably easier proofs, even in the field case. Although we assume, for the sake of simplicity, that our rings are connected, an extension to the non-connected case presents no difficulties because of [8, Corr. 2. 18]. In Section 4, we reprove a special case of [1, Satz 2.7] for the reader's convenience.

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2. *Equivalence of WAP and SAP*

We begin with the following lemma which is already implicit in, i.a., [4, 7, 8]:

Lemma 2.1. *Let R be a connected semilocal ring with 2 in $U(R)$, (E_i, B_i) $i=1, 2$, two spaces, and $r_i=\operatorname{rank} E_i$. If $r_1 \geqq r_2$, and for all σ in X, $\bar{\sigma}([E_1])=\bar{\sigma}([E_2])$, then there is a natural number m such that*

$$2^m E_1 \cong 2^m E_2 \perp 2^{m-1}(r_1-r_2)\mathbf{H},$$

where $\mathbf{H}$ *denotes the hyperbolic plane* $\langle 1, -1\rangle$.

PROOF. By [8, Remark 2.2], the element $[E_1]-[E_2]$ of $W(R)$ lies in each minimal prime ideal of $W(R)$. Hence by [7, Ex. 3. 11] it is a torsion element of $W(R)$, i.e., for some natural number m we have $[2^m E_1]=[2^m E_2]$. Since 2 is in $U(R)$, the Witt Cancellation Theorem holds for R[11], and so the result follows.

As usual a space (E, B) is called an n-Pfister space if $E \cong \bigotimes_1^n \langle 1, a_i\rangle$, a_i in $U(R)$. Two n-Pfister spaces E, F are *linked* [3, Def. 4.1], if there is an $(n-1)$-Pfister space G and elements a, b in $U(R)$, such that $E \cong \langle 1, a\rangle \otimes G$ and $F \cong \langle 1, b\rangle \otimes G$. The spaces E and F are *stably linked* if $2^m E$ and $2^m F$ are linked for some natural number m.

Theorem 2.2. (cf. [4, Thm. 3.5]). *Let R denote a connected semilocal ring with 2 in $U(R)$ and $X=X(R)$ its space of signatures. Then the following are equivalent:*

(1) X satisfies WAP.

(2) X satisfies SAP.

(3) For any n-Pfister space E, there is a natural number m and an element a of $U(R)$ such that $2^m E \cong 2^{m+n-1}\langle 1, a\rangle$.

(4) Any n-Pfister space is stably linked to $2^n\langle 1\rangle$.

(5) Any two n-Pfister spaces are stably linked.

PROOF. We begin by noting that, in order to prove that X satisfies SAP, it suffices to prove that given $a_1, \dots, a_n$ in $U(R)$. There is an element a in $U(R)$ such that $\bigcap_1^n W(a_i) = W(a)$. For, since the $W(a)$'s always constitute a subbase of the topology of X, this shows that they actually form a basis. Then, if Y is an arbitrary clopen subset of X, the clopen set $X-Y$ is a union of $W(a)$'s. Since $X-Y$ is a closed subset of the compact Hausdorff space X, it is itself compact, and thus there exist $a_1, \dots, a_n$ in $U(R)$ with $X-Y = W(a_1)\cup W(a_2)\cup\dots\cup W(a_n)$. Now from the definition, $W(-a) = X - W(a)$, and so $Y = \bigcap_1^n W(-a_i) = W(a)$ for some a in $U(R)$, once the initial statement is proved.

(1)⇒(2).*) For any $a_1, \dots, a_n$ in $U(R)$, let $Y = \bigcap_1^n W(a_i)$. Then Y is open and compact and since X satisfies WAP, there are elements $b_1, \dots, b_k$ in $U(R)$ with $Y = W(b_1)\cup\dots\cup W(b_k)$. By repeating some of the $W(a_i)$'s or $W(b_i)$'s, if necessary, we may suppose

$$Y = \bigcap_1^n W(a_i) = \bigcup_1^n W(b_i),\ a_i, b_i \quad \text{in} \quad U(R).$$

Let $E = \bigotimes_1^n \langle 1, -a_i\rangle$, $F = \bigotimes_1^n \langle 1, b_i\rangle$. Then

if σ is in Y, we have $\bar\sigma([E]) = 2^n$, $\bar\sigma([F]) = 0$,

and

if σ is in $X-Y$, we have $\bar\sigma([E]) = 0$, $\bar\sigma([F)] = 2^n$.

Hence $E \perp F$ and $2^n\langle 1\rangle$ satisfy the conditions of Lemma 2.1, so that for some natural number m,

$$2^m E \perp 2^m F \cong 2^{m+n}\langle 1\rangle \perp 2^{m+n-1}\mathbf{H}.$$

Hence $2^m E \perp 2^m F$ contains a primitive isotropic element. By [1, Satz 2.7(c)] or Proposition 4.1(i), there is an element a in $U(R)$, such that $2^m E$ represents $-a$ while $2^m F$ represents a. If R is a field, this is quite clear, without any reference.

Let $2^m E = \langle c_1, \dots, c_{2^{n+m}}\rangle$. Since for σ in Y, we have $\bar\sigma([2^m E]) = 2^{m+n}$, necessarily $\sigma(c_i) = 1$ for $i = 1, 2, \dots, 2^{m+n}$. Hence by [8, Lemma 2.3(ii)], $\sigma(-a) = 1$, i.e., $\sigma(a) = -1$ for all σ in Y. A similar argument based on $2^m F$ shows that for all σ in $X-Y$ we have $\sigma(a) = 1$. Thus $Y = W(a)$.

*) In case R is a field, essentially the same proof of this implication was also found by T. C. CRAVEN.

$(2)\Rightarrow(3)$. Let $E\cong\bigotimes_{1}^{n}\langle 1, a_i\rangle$ be an n-Pfister space and $Y=W(a_1)\cup W(a_2)\cup\ldots\cup\ldots\cup W(a_n)$. Then $\bar{\sigma}([E])=0$ for σ in Y and $\bar{\sigma}([E])=2^n$ for σ in $X-Y$. Since X satisfies SAP, there is an element a in $U(R)$ with $Y=W(a)$ and clearly the equation $\bar{\sigma}([E])=\bar{\sigma}([2^{n-1}\langle 1, a\rangle])$ holds for all σ in X. Lemma 2.1 then proves (3).

The implication $(3)\Rightarrow(4)$ is clear.

$(4)\Rightarrow(5)$. If E and F denote two n-Pfister spaces, (4) guarantees the existence of elements a, b, c, d in $U(R)$, natural numbers m, m', an $(m+n-1)$-Pfister space G, and an $(m'+n-1)$-Pfister space H, such that

$$2^m E\cong\langle 1, a\rangle\otimes G,\ 2^{m+n}\langle 1\rangle\cong\langle 1, b\rangle\otimes G,$$

$$2^{m'} F\cong\langle 1, c\rangle\otimes H,\ 2^{m'+n}\langle 1\rangle\cong\langle 1, d\rangle\otimes H.$$

Since for all σ in X, we have $\bar{\sigma}(2^{m+n}\langle 1\rangle)=2^{m+n}$, Lemma 2.1 proves the existence of natural numbers k, k' with

$$2^k G\cong 2^{m+n+k-1}\langle 1\rangle,\ 2^{k'} H\cong 2^{m'+n+k'-1}\langle 1\rangle.$$

But then it is clear that $2^{m+k+m'+k'}E$ and $2^{m+k+m'+k'}F$ are linked, proving (5).

$(5)\Rightarrow(2)$. As already remarked, it suffices to show that for $a_1, \ldots, a_n$ in $U(R)$, there is an a in $U(R)$ with $Y=\bigcap_{1}^{n} W(a_i)=W(a)$. Let $E=\bigotimes_{1}^{n}\langle 1, -a_i\rangle$ and $F=2^n\langle 1\rangle$. By (5), there is a natural number m, an $(n+m-1)$-Pfister space G, and elements b, c in $U(R)$ with

$$2^m E\cong\langle 1, b\rangle\otimes G,\ \ 2^m F\cong\langle 1, c\rangle\otimes G\cong 2^{m+n}\langle 1\rangle.$$

Thus for all σ in X, we have $\bar{\sigma}([G])=2^{m+n-1}$. But

for σ in Y, we have $\bar{\sigma}([2^m E)]=2^{m+n}$, and

for σ in $X-Y$, we have $\bar{\sigma}([2^m E])=0$.

It follows, therefore, that $\sigma(b)=1$ for σ in Y, and $\sigma(b)=-1$ for σ in $X-Y$. Hence $Y=W(-b)$, and the proof of Theorem 2.2 is complete since the implication $(2)\Rightarrow(1)$ is clear.

Remark 2.3. In [7, Def. 3.12] the notion of an abstract Witt ring for any abelian p-primary gronp G was introduced. Conditions (2)—(5) are still equivalent for a "small" [8, Rem. 3.17] abstract Witt ring, if G is a group of exponent 2. The n-Pfister spaces are replaced by elements $\prod_{1}^{n}(1+\bar{g}_i)$ of the abstract Witt ring, the group $U(R)$ is replaced by G, the sets $W(a)$ by the sets $W(\bar{g})$ as defined in [8, Thm. 3.18 (ii)], and the isometries in (3), (4), and (5) by equalities in the small abstract Witt ring. However, Craven [2, Section 3] has shown that the implication $(1)\Rightarrow(2)$ cannot hold in such abstract Witt rings, so that it is not surprising that this part of the proof uses facts about units of R represented by spaces.

3. *Equivalence of HMP and SAP*

Theorem 3.1. (cf. [5, Thm. C and 10, Satz 3.1]). *Let R be a connected semilocal ring with 2 in $U(R)$, then SAP is equivalent to HMP.*

PROOF. Let E be a space of rank n over R, and suppose $|\bar{\sigma}([E])|<n$ for all σ in X. Then the possible values of $\bar{\sigma}([E])$ are $-n+2, -n+4, \ldots, n-4, n-2$. Let $Y_k=\{\sigma \text{ in } X|\sigma([E])=-n+2k\}$, $k=1, 2, \ldots, n-1$. Clearly the Y_k are a partition of X and, of course, some Y_k may be empty. Now, the function $f_E\colon X\to\mathbf{Z}$ defined by $f_E(\sigma)=\bar{\sigma}([E])$ is continuous if $\mathbf{Z}$ is given the discrete topology [8, Lemma 3.3 (iv)], so that each Y_k is a clopen subset of X. Since X satisfies SAP, there are elements $b_1, b_2, \ldots, b_{n-2}$ in $U(R)$ such that $W(b_1)=Y_1$, $W(b_2)=Y_1\cup Y_2, \ldots, W(b_{n-2})=$ $=Y_1\cup Y_2\cup\ldots\cup Y_{n-2}$. Now let $F\cong\langle b_1, \ldots, b_{n-2}\rangle$ and σ be an element of Y_k, $k=1, 2, \ldots, n-2$. Since Y_k is contained in $W(b_k), W(b_{k+1}), \ldots, \ldots, W(b_{n-2})$, but is disjoint from $W(b_1), W(b_2), \ldots, W(b_{k-1})$, we have $\bar{\sigma}([F])=(k-1)-(n-2-k+1)=$ $=-n+2k=\bar{\sigma}([E])$. If σ lies in Y_{n-1}, then since Y_{n-1} is disjoint from all $W(b_i)$, we still have $\bar{\sigma}([F])=n-2=\bar{\sigma}([E])$. Since the Y_i's cover X, Lemma 2.1 guarantees the existence of a natural number m such that $2^mE\cong 2^mF\perp 2^m\mathbf{H}$. Hence 2^mE is isotropic, and R satisfies HMP.

HMP$\Rightarrow$SAP. As at the beginning of the proof of Theorem 2.2, it suffices to prove that for any two elements a, b of $U(R)$, there is an element c in $U(R)$ with $W(a)\cap W(b)=W(c)$. Thus, for a, b in $U(R)$, let $E\cong\langle -1, a, b, ab\rangle$. It is easily verified that for all σ in X, we have $\bar{\sigma}([E])=\pm 2$, so that by HMP, there is a natural number m such that 2^mE is isotropic. By multiplying by 2, if necessary, we assume $m\geqq 1$. Thus $2^m\langle -1\rangle\perp 2^m\langle a, b, ab\rangle$ is isotropic and so again by [1, Satz 2.7 (c)] or Proposition 4.1 (i), there is an element t in $U(R)$ such that $-t$ is represented by $2^m\langle -1\rangle$ and t is represented by $2^m\langle a, b, ab\rangle$. Thus for some r_i in R, $t=\sum r_i^2$, so that by [8, Lemma 2.3 (ii)], $\sigma(t)=1$ for all σ in X. Now t is represented by $2^m\langle a\rangle\perp\langle b\rangle\otimes$ $\otimes 2^m\langle 1, a\rangle$ and by [1, Satz 2.7 (a)] or Prop. 4.1 (ii), there are units s and v in R such that $t=s+v$, s is represented by $2^m\langle a\rangle$, and v is represented by $\langle b\rangle\otimes 2^m\langle 1, a\rangle$. Setting $c=b^{-1}v$ we then have $t=s+bc$, with c represented by $2^m\langle 1, a\rangle$. Once more [8, Lemma 2.3 (ii)] shows $W(s)=W(a)$ and $W(c)\subset W(a)$. Now let σ lie in $W(c)\subset W(a)$ and suppose $\sigma(b)=1$. Then $\sigma(bc)=-1=\sigma(a)$, which again by [8, Lemma 2.3 (ii)] would force $\sigma(t)=-1$, a contradiction. Hence $W(c)\subset W(a)\cap W(b)$. On the other hand, if σ lies in $W(a)\cap W(b)$, then $\sigma(s)=-1$ and so, since $\sigma(t)=$ $=1$, we again see by [8, Lemma 2.3 (ii)] that $\sigma(bc)=1$, but this means $\sigma(c)=-1$, or $W(a)\cap W(b)\subset W(c)$, i.e., $W(a)\cap W(b)=W(c)$ as desired.

4. *Representations of units*

The following Proposition is a special case of [1, Satz 2.7], and is only given here for the reader's convenience.

Proposition 4.1. *Let R be a connected semilocal ring with 2 in $U(R)$ and (E, B) a space over R. Let $E=E_1\perp E_2$.*

(i) *If the rank of E is $\geqq 3$ and e is a primitive isotropic element of E, there existr a primitive isotropic element e' of E such that $e'=e_1'+e_2'$, e_i' in E_i, and $B(e_i', e_i')$ lies in $U(R)$.*

(ii) *If, in addition, rank* $E_i \geqq 2$, $i=1,2$, *and* e *is an element of* E *with* $B(e,e)$ *in* $U(R)$, *then there exists an element* e' *in* E *with* $e'=e_1'+e_2'$, e_i' *in* E_i, *such that* $B(e,e)=B(e',e')$ *and* $B(e_i',e_i')$ *lies in* $U(R)$.

PROOF. We first prove (i) and (ii) in case R is a field, necessarily of characteristic different from 2. Since $E=E_1\perp E_2$, we have $e=e_1+e_2$, with e_i in E_i, and if both $B(e_1,e_1)$ and $B(e_2,e_2)$ are different from zero we may take $e'=e$, $e_i'=e_i$. Hence, by renumbering, if necessary, we may suppose $B(e_1,e_1)=0$, i.e., E_1 is isotropic. Since E is nondegenerate, E_1 and E_2 are also, and so E_1 is universal. Since E_2 is nondegenerate and the characteristic of R is not 2, there is a vector e_2' in E_2 with $B(e_2',e_2')\neq 0$. To complete (i), we simply choose e_1' in E_1 with $B(e_1',e_1')=-B(e_2',e_2')$.

In case (ii), let $B(e_2,e_2)=B(e,e)=a\neq 0$. Suppose first that R is not the field of 3 elements. Since R contains at least three nonzero elements, R contains a nonzero element $b\neq\pm a$, and since E_1 is universal there is a vector $\tilde{e}_1$ in E_1 with $B(\tilde{e}_1,\tilde{e}_1)==b$. Now there is a nonzero element c in $R, c\neq\pm 1$, such that $b=ca$. Since $c=\left(\frac{c+1}{2}\right)^2-\left(\frac{c-1}{2}\right)^2$, we have $c=u^2-v^2$ with u, v nonzero elements of R. Thus $b=(u^2-v^2)a$, or $a=\frac{b}{u^2}+\frac{v^2}{u^2}a$, and setting $e_1'=\frac{\tilde{e}_1}{u}$, $e_2'=\frac{v}{u}e_2$, $e'=e_1'+e_2'$, completes the proof if R is a field of more than three elements. If R is the field of three elements, then $2a+2a=a$. But over R any space of rank 2 is universal [9, 62. 1, p. 157] and thus there exist e_i' in E_i with $B(e_i',e_i')=2a$.

Still supposing R a field, a classical theorem due to Witt [9, 42.17, p. 98], ensures that there is an isometry T of E such that $T(e)=e'$. Now there is a vector f in E orthogonal to e and such that $B(f,f)\neq 0$. For, if dim $E=n$, then dim $(\mathrm{Re})^\perp=n-1$, and if $(\mathrm{Re})^\perp$ were totally isotropic then $n-1\leqq\frac{n}{2}$, which forces $n\leqq 2$. Thus, if necessary, multiplying T on the right by the reflection $x\to x-\frac{2B(x,f)}{B(f,f)}f$, we may assume that det $T=+1$.

To settle the case where R is a semilocal ring, let $O^+(E)$ denote the group of those isometries of E whose images on E/m_jE have determinant $+1$ for all maximal ideals m_j, $j=1,\dots,k$, of R. By the above, there exist vectors $\bar{e}_j'$ in E/m_jE with the desired projections on E_i/m_jE_i, $i=1,2$, and elements T_j in $O^+(E/m_jE)$, such that $T_j(e+m_jE)=\bar{e}_j'$. Now KNEBUSCH has proved that the natural map $O^+(E)\to\to\prod_1^k O^+(E/m_jE)$ is surjective [6, Satz 0.4]. Hence if T is an element in $O^+(E)$ mapping onto $(T_1,T_2,\dots,T_k)$ in $\prod_1^k O^+(E/m_jE)$, it is clear that $e'=T(e)$ and the projections of e' into E_1 and E_2, have the desired properties.

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