

Investigations in the powersum theory V.

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To the memory of A. Kertész

1. In our paper [1] to appear we proved as a first step to a general theorem the following.

Theorem A. *Let us consider the equation*

$$(1.1) \quad Y^{(n)} + a_{n-1} Y^{(n-1)} + \dots + a_0 = 0, \quad Y = Y(t), a_v \text{ real constans.}$$

We suppose that the equation

$$(1.2) \quad f(z) \doteq a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0 \quad (a_n = 1)$$

has no zeros in the strip

$$(1.3) \quad |\operatorname{Im} z| \leq \Lambda$$

Then all real solutions $Y(t) \neq 0$ change sign in every real interval of length

$$(1.4) \quad > \frac{n\pi}{2\Lambda}.$$

This is best possible for all $n \geq 2$ and $\Lambda > 0$.

An analogous result could have been proved for the difference equation

$$(1.5) \quad x_{n+l} + b_{n-1} x_{n+l-1} + \dots + b_1 x_{l+1} + b_0 x_l = 0, \quad l = 0, 1, \dots$$

2. The proof was based on a "onesided" powersum theorem (see [2]) which proved to be useful in several situations e.g. in analytical number theory. We shall not formulate it here but mention that in its proof the following auxiliary problem played an essential role. Let $\pi_n(x)$ be a polynomial of n^{th} degree with $\pi_n(0) = 1$ say so that all of its roots are outside the angle

$$(2.1) \quad |\arg z| \leq \varkappa$$

with a $0 < \varkappa \leq \frac{\pi}{2}$. One has to multiply it by a polynomial $q_k(x)$ with $q_k(0) = 1$ and possibly small degree k so that all coefficients of $\pi_n(x)q_k(x)$ are nonnegative. In

order to get (1.4) we had to solve in our paper IV. the corresponding problem when in addition the restriction

$$(2.2) \quad \text{the coefficients of } \pi_n(x) \text{ are real}$$

is satisfied. In the present paper — which is strictly speaking a digression — we are going to show that a “theorem of alternative” equivalent to the Farkas theorem from the theory of linear inequalities gives possibility to go from the abovementioned “multiplication problem” over *directly* to the result (1.5) and then after a simple passage to limit to theorem A. Calling a vector X in R^n positive, $X > 0$ when all coordinates are positive the theorem of alternative in question* asserts that if G is a matrix with real elements then exactly one of the inequalities

$$(2.3) \quad GX \geq 0, \quad GX \neq 0 \quad (X \text{ column vector})$$

$$(2.4) \quad YG = 0, \quad Y > 0 \quad (Y \text{ row vector})$$

is solvable.

3. In order to prove theorem A on this way we shall make first a detour and consider the product $\varphi(z)q_k(z)$ where

$$(3.1) \quad \varphi(z) = d_0 + d_1 z + \dots + d_n z^n,$$

$$g_k(z) = x_0 + x_1 z + \dots + x_k z^k,$$

and the x_v 's are real variables. The coefficients of the product, if $X = (x_0, x_1, \dots, x_k)$, form a columnvector which can be written as

$$(3.2) \quad DX$$

where D is an $(n+k+1) \times (k+1)$ matrix whose explicit form for $k \leq n$ resp. $k > n$ is

$$D = \begin{pmatrix} d_0 & 0 & \dots & 0 \\ d_1 & d_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_k & d_{k-1} & \dots & d_0 \\ d_{k+1} & d_k & \dots & d_1 \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n-1} & \dots & d_{n-k} \\ 0 & d_n & \dots & d_{n-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

*) See J. STOER and C. WITZGALL, Convexity and Optimization in Finite Dimensions I, Berlin, 1970, p. 23.

resp.

$$D = \begin{pmatrix} d_0 & 0 & \dots & & & 0 \\ d_1 & d_0 & 0 & \dots & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ d_n & d_{n-1} & \dots & d_0 & 0 & \dots & 0 \\ 0 & d_n & \dots & d_1 & d_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & & d_n & d_{n-1} & \dots & d_0 \\ \vdots & & & d_n & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & 0 & d_n \end{pmatrix}$$

We define $k_0 = k_0(\varphi)$ as the minimum of integer k 's for which there is a suitable $q_k^*(z) \not\equiv 0$ so that all coefficients of the product are nonnegative; if there is no such k (when $\varphi(z)$ has a positive zero of odd multiplicity e.g.) then let $k_0 = \infty$. We may suppose $k_0 \equiv 1$. Choosing k in (3.1) as $(k_0 - 1)$ the system

$$(3.3) \quad DX \equiv 0, \quad DX \not\equiv 0$$

is *not* solvable. But then from the above quoted theorem of alternative the system

$$(3.4) \quad YD = 0, \quad Y > 0$$

is solvable. If

$$(3.5) \quad Y = (y_0, y_1, \dots, y_{n+k_0})$$

is such a solution then we have

$$(3.6) \quad y_v > 0 \quad (v = 0, 1, \dots, (n+k_0))$$

and the relations

$$\begin{aligned} d_0 y_0 + d_1 y_1 + \dots + d_n y_n &= 0 \\ d_0 y_1 + d_1 y_2 + \dots + d_n y_{n+1} &= 0 \\ \vdots & \\ d_0 y_{k_0} + d_1 y_{k_0+1} + \dots + d_n y_{n+k_0} &= 0 \end{aligned}$$

This means owing to $a_n = 1$ that the difference equation

$$(3.7) \quad d_n y_{m+n} + d_{n-1} y_{m+n-1} + \dots + d_0 y_m = 0$$

$$m \equiv 0$$

has a solution which is positive for

$$(3.8) \quad 0 \leq m \leq k_0 + n.$$

Hence if m_0 stands for the maximal length of an interval where a solution of the equation (3.7) can be positive then we got

$$(3.9) \quad m_0 \geq k_0 + n.$$

4. Next we start with a solution y_m^* of the equation (3.7) which is positive for an interval of length m_0 . This means that the system

$$(4.1) \quad \begin{aligned} d_0 y_0^* + d_1 y_1^* + \dots + d_n y_n^* &= 0 \\ d_0 y_1^* + d_1 y_2^* + \dots + d_n y_{n+1}^* &= 0 \\ \dots &\dots \\ d_0 y_{m_0-n}^* + d_1 y_{m_0-n+1}^* + \dots + d_n y_{m_0}^* &= 0 \end{aligned}$$

has a solution with

$$(4.2) \quad y_j^* > 0 \quad (j = 0, 1, \dots, m_0).$$

But this means that the system

$$YD_1 = 0, \quad Y > 0, \quad Y \text{ row vector}$$

is solvable by $Y = Y^*$ where D_1 stands for the $(m_0 - n + 1) \times (m_0 + 1)$ matrix

$$(4.3) \quad D_1 = \begin{pmatrix} d_0 & 0 & 0 & \dots & 0 \\ d_1 & d_0 & & \dots & 0 \\ \vdots & & & & \vdots \\ d_n & d_{n-1} & & \dots & \\ 0 & d_n & & \dots & d_0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \dots & 0 & d_n \end{pmatrix}$$

The application of the theorem of alternative gives then the *non* solvability of the system

$$D_1 X \geq 0, \quad D_1 X \not\equiv 0.$$

Hence if k_0 has the previous meaning then

$$(4.4) \quad k_0 + 1 \geq m_0 - n + 1.$$

This and (3.9) give the

Theorem I. For the above defined m_0 and k_0 the relation

$$m_0 = k_0 + n$$

holds.

This theorem reduces the determination of maximal length of a positivity-interval of an arbitrary solution of the difference equation (3.7) to the determination of the minimal $k = k_0$ for which a $q_k(z) \not\equiv 0$ polynomial exists so that the product

in (3.1) has nonnegative coefficients. Since the following theorem of alternative also holds

$$(4.5) \quad GX > 0$$

resp

$$(4.6) \quad YG = 0, \quad Y \geq 0, \quad Y \neq 0$$

the repetition of the previous reasoning gives

Theorem II. *The relation*

$$m_1 = k_1 + n \quad Y \neq 0$$

holds if m_1 stands for the maximal length of nonnegativity interval of an arbitrary solution of the difference equation (3.7) and k_1 is the minimal k for which a $q_k(z)$ polynomial exists so that the product in (3.1) has positive coefficients.

Obviously if we know upper bounds for $k_0 = k_0(\varphi)$ and $k_1 = k_1(\varphi)$ then we get upper bounds for m_0 resp. m_1 .

5. It was shown in [2] in the important case

$$(5.1) \quad \varphi^*(z) = r^2 - 2 \cos \alpha rz + z^2$$

$$r > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

the inequality

$$k_0(\varphi^*) \leq \left[\frac{\pi}{|\alpha|} \right] - 1$$

and in [3] where for the first time the general theory of inequalities was applied to that sort of equations that

$$(5.2) \quad k_0(\varphi^*) = \begin{cases} \frac{\pi}{|\alpha|} - 2 & \text{if } \frac{\pi}{|\alpha|} \text{ is an integer,} \\ \left[\frac{\pi}{|\alpha|} \right] - 1 & \text{if } \frac{\pi}{|\alpha|} \neq \text{integer.} \end{cases}$$

In the general case (1.2) $f(z)$ is a product of factors of form

$$z + r_v \quad r_v > 0$$

and also of the form

$$(5.3) \quad \begin{aligned} & z^2 - 2 \cos \alpha_v r_v z + r_v^2 \\ & r_v > 0, \quad 0 < |\alpha_v| \leq \pi. \end{aligned}$$

We have to care only with factors from (5.3) with

$$0 < |\alpha_v| < \frac{\pi}{2};$$

applying (5.2) to each them we get the general inequality

$$(5.4) \quad k_0(\varphi) \leq \sum' \left(\frac{\pi}{|\alpha_v|} - 2 \right) + \sum'' \left(\left\lfloor \frac{\pi}{|\alpha_v|} \right\rfloor - 1 \right)$$

where $\sum'$ resp. $\sum''$ is the extended to the α_v 's with

$$\frac{\pi}{|\alpha_v|} = \text{integer} \geq 3,$$

resp.

$$\frac{\pi}{|\alpha_v|} > 2 \quad \text{and} \neq \text{integer}.$$

Suppose now that for $\varphi(z)$ with real coefficients there is a κ with

$$(5.5) \quad 0 < \kappa < \frac{\pi}{2}$$

so that $\varphi(z) \neq 0$ for

$$(5.6) \quad |\arg z| \leq \kappa$$

Then (5.4) and (5.6) give at once

Theorem III. *If the polynomial $\varphi(z)$ of n^{th} degree and real coefficients satisfies (5.6) then the inequality*

$$k_0(\varphi) \leq \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{\pi}{\kappa} \right\rfloor - 1 \right) \leq \frac{n}{2} \left(\left\lfloor \frac{\pi}{\kappa} \right\rfloor - 1 \right)$$

holds.

Combining it with theorem I. we get

Theorem IV. *If*

$$(5.7) \quad \varphi(z) = d_0 + d_1 z + \dots + d_n z^n \quad (d_n = 1)$$

with real d_v 's and (5.6) is satisfied then no solution y_l of the difference equation

$$(5.8) \quad d_0 y_l + d_1 y_{l+1} + \dots + d_n y_{l+n} = 0$$

can be positive for more than

$$(5.9) \quad \frac{n}{2} \left(\left\lfloor \frac{\pi}{\kappa} \right\rfloor + 1 \right)$$

consecutive integer l -values.

6. Next we show that equality in (5.9) can be attained for each $0 < \kappa < \frac{\pi}{2}$ and even n . Let namely $n = 2N$ and the coefficients d_v^* in (5.7) be defined by

$$(6.1) \quad d_0^* + d_1^* z + \dots + d_{2N}^* z^{2N} = (1 - 2 \cos \alpha z + z^2)^N,$$

where $0 < \alpha < \frac{\pi}{2}$ is such that $\frac{\pi}{\alpha}$ is not an integer (an unessential restriction). Then we have

$$\kappa = \alpha$$

of course. Then all sequences y_l^{**} of the form

$$(6.2) \quad y_l^{**} = (c_0 + c_1 l + \dots + c_{N-1} l^{N-1}) \sin(\alpha l + \vartheta)$$

ϑ , c_v real constants, l variable satisfy the equation

$$(6.3) \quad d_0^* y_l + d_1^* y_{l+1} + \dots + d_{2N}^* y_{l+2N} = 0$$

Let ϑ be any number with

$$0 < \vartheta < \pi - \alpha \left\lceil \frac{\pi}{\alpha} \right\rceil$$

for which in addition none of the numbers

$$\frac{v\pi - \vartheta}{\alpha}, \quad v = 0, \pm 1, \dots$$

are integers. Fixing such a $\vartheta = \vartheta_0$ and writing

$$\frac{v\pi - \vartheta_0}{\alpha} = \beta_v$$

we see at once that the sequence

$$y_l^{**} = \prod_{v=1}^{N-1} (\beta_v - l) \sin(\alpha l + \vartheta_0)$$

is of form (6.2) and

$$y_l^{**} > 0 \quad \text{for } l = 0, 1, \dots, N \left(\left\lceil \frac{\pi}{\alpha} \right\rceil + 1 \right) - 1$$

indeed.

7. In order to deduce theorem A from theorem IV. we reformulate first theorem IV. If $z_1, z_2, \dots, z_k$ are different zeros of the equation (5.7) then all solutions of equation (5.8) have the form

$$(7.1) \quad y_l = \sum_{\mu=1}^k Q_{\mu}(l) z_{\mu}^l$$

Where complex z_{μ} 's occur only in conjugate pairs further the $Q_{\mu}(l)$'s are polynomials in l so that

$$(7.2) \quad \sum_{\mu=1}^k \deg Q_{\mu}(x) = n - k$$

and $Q_\mu(x)$'s belonging to complex-conjugate z_μ have complex-conjugate coefficients and conversely, each such sequence (7.1)—(7.2) satisfies an (essentially uniquely determined) equation (5.8). Thus theorem IV. means that all sequences of type (7.1)—(7.2) with no z_j 's in the angle

$$(7.3) \quad |\arg z| \leq \varkappa$$

are such that no consecutive L terms can be positive if only

$$(7.4) \quad L > \frac{n}{2} \left(\left\lceil \frac{\pi}{\varkappa} \right\rceil + 1 \right).$$

8. Now we deduce quickly theorem A. Let y be an arbitrary solution of (1.1) where (1.2)—(1.3) satisfied. Denoting the zeros of (1.2) by ξ_j ($j=1, 2, \dots, n$) we have

$$(8.1) \quad Y(t) = \sum_{\mu=1}^k P_\mu(t) e^{\xi_\mu t}$$

where $\xi_1, \dots, \xi_\mu$ are all pairwise different zeros of (1.2), the $P_\mu(t)$'s are polynomials with

$$(8.2) \quad \sum_{\mu=1}^k \deg P_\mu(t) = n - k.$$

The restrictions made imply that $Y(t)$ is real for real t . If a is an arbitrary real number and h is sufficiently small, both fixed then let us consider the sequence

$$(8.3) \quad y(l) \stackrel{\text{def}}{=} Y(a+hl) = \sum_{\mu=1}^k P_\mu(a+hl) e^{\xi_\mu a} (e^{\xi_\mu h})^l.$$

Suppose now that for $\mu=1, 2, \dots, n$

$$(8.4) \quad \min_{\mu} |\operatorname{Im} \xi_\mu| = \Lambda > 0$$

and we apply theorem IV. as reformulated in 7. with

$$(8.5) \quad z_\mu = e^{\xi_\mu h}, \quad Q_\mu(l) = P_\mu(a+hl) \quad \mu = 1, 2, \dots, k.$$

Then if h is sufficiently small then we can take as $\varkappa$ in (7.3)

$$(8.6) \quad \varkappa = h\Lambda.$$

Hence owing to (7.4) there are integers l_1 and l_2 with

$$0 \leq l_1 \neq l_2 \leq L_1$$

and with

$$L_1 = \frac{n}{2} \left(\left\lceil \frac{\pi}{h\Lambda} \right\rceil + 1 \right) + c, \quad c > 0$$

so that

$$y(l_1) > 0, \quad y(l_2) \leq 0.$$

Since

$$a \leq a + hl_1 \leq a + \frac{n}{2} \left(\frac{\pi}{A} + h \right) + ch,$$

$$a \leq a + hl_2 \leq a + \frac{n}{2} \left(\frac{\pi}{A} + h \right) + ch,$$

this means that

$$\max_{t \in I} Y(t) > 0 \quad \min_{t \in I} Y(t) \leq 0$$

where

$$I = \left[a, a + \frac{n\pi}{2A} + \left(\frac{n}{2} + c \right) h \right].$$

Thus for $h \rightarrow 0$ we obtained again theorem A.

References

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