

A certain radius of convexity problem

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Introduction. Let E denote the unit disc and C denote the class of functions f regular in E normalised by $f(0)=f'(0)-1=0$ and which satisfies the condition that it maps E onto a domain which is convex. It is interesting to ask whether

$$f \text{ and } g \in C \text{ imply } \lambda f + (1-\lambda)g \in C \text{ for } 0 < \lambda < 1.$$

(Cf. Problem 6.11 in [2]). This problem could be answered in the negative. However the radius of convexity of the family consisting of $\lambda f + (1-\lambda)g$ for arbitrary $f, g \in C$ and each fixed $\lambda \in (0, 1)$ is still not known. A more general problem would be to ask for the radius of convexity of $\lambda f + (1-\lambda)g$ where f is convex and g is starlike, (with respect to origin). Since a complete solution is not known, it will be interesting to consider special cases in which the problem admits a complete solution. We can weaken the question either by specialising f or g or both. In this context the choice of $g=zf'$ is interesting, since the starlike function associated to an $f \in C$ is zf' , in the sense that $f \in C$ iff zf' is starlike (with respect to origin). We specialise f also by requiring that $p(z)=[1+(zf''(z)/f'(z))]$ should satisfy

$$|(p(z)-1)(p(z)+1)^{-1}| < \alpha \text{ for } z \in E \text{ and } 0 < \alpha \leq 1.$$

We also choose $\lambda=2^{-1}$.

Both the choices $\lambda=2^{-1}$ and the speciality of f indicated above are motivated by the fact that the result obtained may be a natural refinement of the one obtained by LIVINGSTON in [3]. The technique we adopt is suggested by the work in [4]. Thus in this paper we find out the radius of convexity of the family $C(\alpha)$ which can be described as follows.

$f(z) \in C(\alpha)$ iff f is regular in E and satisfies $f(0)=0, f'(0)=1$ and $|(p(z)-1)(p(z)+1)^{-1}| < \alpha$ where $p(z)=[1+(zf''(z)/f'(z))]$ and then

$$P(\alpha) = \{g(z) = 2^{-1}[f(z) + zf'(z)]/f(z) \in C(\alpha)\}.$$

Theorem. The radius of convexity γ^* of the family $C(\alpha)$ can be described as follows.

- (i) For $0 < \alpha \leq [(\sqrt{5}+1)/4]$, $\gamma^* = [(p-1+\alpha)/2\alpha]^{1/2}$ where $p^2 = (1-\alpha)(1+3\alpha)$.
- (ii) For $[(\sqrt{5}+1)/4] \leq \alpha \leq 1$, $\gamma^* = (2\alpha)^{-1}$.

PROOF. Let $g(z) = 2^{-1}[f(z) + zf'(z)]$, $f(z) \in C(\alpha)$ $p(z) = [1 + (zf''(z)/f'(z))]$, $p^*(z) = [1 + (zg''(z)/g'(z))]$, and $w(z) = (p(z) - 1)\alpha^{-1}(p(z) + 1)^{-1}$, $r = |z|$. From $g \in P(\alpha)$ we have $|w(z)| < 1$ for $z \in E$. $w(0) = 0$ and so by Schwarz lemma $|w(z)| \leq r$ ($z \in E$). Simple calculation shows that

$$(1) \quad \operatorname{Re} p^*(z) = -1 + 2 \operatorname{Re} q(z) - \operatorname{Re} \alpha zw'(z)q(z)$$

where $q(z) = [1 + \alpha w(z)]^{-1}$.

From $|w(z)| \leq r$ we have

$$|q(z) - a| \leq d \quad \text{where} \quad a = (1 - \alpha^2 r^2)^{-1}, \quad d = \alpha r a.$$

Now $\Phi(z) = z^{-1}w(z)$ satisfies $|\Phi(z)| \leq 1$ ($z \in E$) and so by a well known result [1],

$$|\Phi'(z)| \leq (1 - r^2)^{-1}(1 - |\Phi(z)|^2).$$

So, $|zw'(z) - w(z)| \leq (r^2 - |w(z)|^2)(1 - r^2)^{-1}$ and thus $\operatorname{Re} zw'(z)q(z) \leq \operatorname{Re} w(z)q(z) + (r^2 - |w(z)|^2)(1 - r^2)^{-1}|q(z)|$. Using these result and (1) elementary calculations give

$$(2) \quad \operatorname{Re} p^*(z) \leq -2 + 3(a + u) - (1 - \alpha^2 r^2)(1 - r^2)^{-1}[(a + u)^2 + v^2]^{-1/2}(d^2 - u^2 - v^2)$$

where u and v real numbers depending on z , $q(z) = a + u + iv$, $u^2 + v^2 \leq d^2$. If we denote the right side of (2) which is a function of u and v by $s(u, v)$ we find that

$$\partial s(u, v) / \partial v = \alpha^{-1}(1 - r^2)^{-1}(1 - \alpha^2 r^2)vR^{-4}T(u, v)$$

where $R = |q(z)|$ and $T(u, v) > 0$.

So the right side of (2) considered as a function of v with u fixed takes its minimum for $u^2 + v^2 \leq d^2$ at $v = 0$ and so,

$$(3) \quad \operatorname{Re} p^*(z) \leq -2 + 3(a + u) - (1 - \alpha^2 r^2)(1 - r^2)^{-1}\alpha^{-1}(a + u)^{-1}(d^2 - u^2).$$

Now we consider the right side of (3) as a function $F(u)$ of $u \leq -d$, with r fixed and calculate the number $u_0 = u_0(r)$ where F takes its absolute minimum. Since $|u| \leq d$ in (3) and F takes its minimum for $|u| \leq d$ at u_0 , we have,

$$(4) \quad \operatorname{Re} p^*(z) \leq F(u_0) \quad \text{if} \quad u_0 \leq -d.$$

If $u_0 < -d$, F is a monotonic increasing function of u and so,

$$(5) \quad \operatorname{Re} p^*(z) \leq F(-d) \quad \text{if} \quad u_0 < -d.$$

Now g is convex for $|z| < \rho$ if $\operatorname{Re} p^*(z) > 0$ for $|z| < \rho$. Thus we can get (i) and (ii) from (4) and (5) by elementary calculations which can be left to the reader.

We can see (ii) is sharp for each α by considering the function f defined by

$$p(z) = (1 - \alpha z)(1 + \alpha z)^{-1}. \quad \text{In the case of (i),}$$

consider the function f defined by $w(z)=z(z-t)(1-tz)^{-1}$ where t is determined by the condition $1+\alpha w(r_1)=[u_0(r_1)+a]^{-1}$,

$$r_1^2 = (2\alpha)^{-1}(p-1+\alpha).$$

The detailed calculation is again left to the reader.

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