

On radicals in a certain class of semigroups

By HARBANS LAL (New Delhi)

We shall call a semigroup S a σ -reflexive semigroup if any subsemigroup H in S is reflexive (that is, for all a, b in S , $ab \in H \Rightarrow ba \in H$) ([2, 3, 7]). A commutative semigroup is clearly a σ -reflexive semigroup. J. BOŠÁK [1, p. 209] and R. ŠULKA [6, p. 221] proved that if S is a commutative semigroup and J any ideal of S , then the Clifford, McCoy, Ševrin, Schwarz and Luh radicals with respect to J (respectively denoted by $R_J^*(S)$, $M_J(S)$, $L_J(S)$, $R_J(S)$ and $C_J(S)$) are equal to $N_J(S)$, the set of all nilpotent elements of S with respect to J . For their definitions, we refer to [1, 6]. Further J. E. KUCZKOWSKI [4] proved that if S is a C_2 -semigroup, then $M_J(S) = L_J(S) = R_J^*(S) = C_J(S) = N_J(S)$ for every ideal J of S . The purpose of this note is to extend the previous result to the class of σ -reflexive semigroups obtaining the following

Theorem. *If S is a σ -reflexive semigroup and J any ideal of S ; then $R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S)$.*

First we recall the following proposition from [2].

Proposition. *Any semigroup S is σ -reflexive if and only if it satisfies the following condition:*

$$\forall a, b \in S, \exists m = m(a, b) \cong 1; \quad ab = (ba)^m.$$

Before coming to the proof of the theorem, we prove the following two lemmas.

Lemma 1. *Let S be a σ -reflexive semigroup. Then an ideal of S is prime if and only if it is completely prime [1].*

PROOF. Clearly it suffices to prove that any prime ideal is completely prime. Let P be any prime ideal and $ab \in P$, where $a, b \in S$. We shall show that $a \in P$ or $b \in P$. Let x be any arbitrary element of S . Then by the previous proposition, $ax = (xa)^m$, where m is an integer $\cong 1$. Now $axb = (xa)^m \cdot b \in P$ for every x in S , since $ab \in P$, whence $aSP \subseteq P$. As P is prime, we get $a \in P$ or $b \in P$, so that P is a completely prime ideal.

Corollary. Let S be a σ -reflexive semigroup and J any ideal of S . Then $M_J(S) = C_J(S)$.

Let x be any element of a semigroup S . The principal ideal generated by x will be denoted by $J(x)$.

Lemma 2. Let S be a σ -reflexive semigroup. Then for any x, y in S , $J(x) \cdot J(y) = J(xy)$.

PROOF. Clearly $J(xy) \subseteq J(x) \cdot J(y)$. Let $a \in J(x)$ and $b \in J(y)$. Then a is one of $x, t_1 x, xt_2$, or $t_1 xt_2$ and b is one of $y, t_3 y, yt_4$, or $t_3 yt_4$ where $t_i \in S$ for $i = 1, 2, 3, 4$. By making use of the previous proposition, it is a routine verification that $ab \in J(xy)$ in every case. Thus $J(x) \cdot J(y) \subseteq J(xy)$, whence the lemma.

PROOF OF THE THEOREM. J. BOSÁK [1, Theorem 2] proved that

$$R_J(S) \subseteq M_J(S) \subseteq L_J(S) \subseteq R_J^*(S) \subseteq N_J(S) \subseteq C_J(S),$$

for any semigroup S . Since $M_J(S) = C_J(S)$ by the above corollary, we get the equality of five of them barring $R_J(S)$. This will be equal to the others if we show $R_J^*(S) \subseteq R_J(S)$ and this we do now. Let $a \in R_J^*(S)$; then $a^n \in J$ for some positive integer n . Since $[J(a)]^n = J(a^n) \subseteq J$, by Lemma 2, $J(a)$ is a nilpotent ideal with respect to J and hence $a \in R_J(S)$. This completes the proof of the theorem.

Remark. This theorem no more remains true if we drop from its hypothesis the σ -reflexivity of the semigroup. For instance, let $S = \{0, a_1, a_2, a_3, a_4\}$ be a semigroup with the following multiplication table:

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	0	a_3	0
a_2	0	0	a_2	0	a_4
a_3	0	0	a_3	0	a_1
a_4	0	a_4	0	a_2	0

Here $H = \{0, a_2, a_4\}$ is a subsemigroup of S such that $a_3 a_1 (= 0)$ is in H , whereas $a_1 a_3 (= a_3)$ is not in H ; whence S is not σ -reflexive. Let $J = \{0\}$; then $R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = \{0\}$; $N_J(S) = \{0, a_3, a_4\}$ and $C_J(S) = S$. This example is due to Š. SCHWARZ [5].

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HANS RAJ COLLEGE,
DELHI UNIVERSITY,
DELHI-7, INDIA.

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