

On derivatives of an algebraic polynomial of best approximation with weight

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Let us denote by $L_p(-\infty, \infty)$ — as usual — the Banach space of measurable functions in $(-\infty, \infty)$ with norm

$$\|f(x)\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p dx \right\}^{1/p} \quad (1 \leq p < \infty)$$

resp. for $p = \infty$,

$$\|f(x)\|_{\infty} = \text{vrai} \cdot \sup_{-\infty < x < \infty} |f(x)|.$$

Let P_n be a set of algebraic polynomials of degree not greater than n ($n=0, 1, \dots$).
Let

$$(1) \quad v_{\alpha, \beta}(x) = (1 + |x|^{\alpha})^{\beta/2\alpha} e^{-|x|^{\alpha/2}}, \\ -\infty < x < \infty \quad \alpha \geq 2, \quad \beta \geq 0.$$

For an f satisfying $v_{\alpha, \beta} f \in L_p(-\infty, \infty)$, we define

$$(2) \quad E_n = E_n^{(p)}(v_{\alpha, \beta}; f) = \inf_{q \in P_n} \|v_{\alpha, \beta}(f - q)\|_p \quad n = 0, 1, \dots$$

We have for each $v_{\alpha, \beta} f \in L_p(-\infty, \infty)$,

$$(3) \quad E_n^{(p)}(v_{\alpha, \beta}; f) \rightarrow 0 \quad (n \rightarrow \infty).$$

It is known that there exists a unique

$$(4) \quad q_n(x) = q_n(f; x) = q_n(L_p; f; x) \quad \text{that} \quad E_n^{(p)}(v_{\alpha, \beta}; f) = \|v_{\alpha, \beta}(f - q_n)\|_p.$$

The polynomial $g_n(L_p; f; x)$ is called the n -th polynomial of best approximation of $f(x)$ with the weight $v_{\alpha, \beta}(x)$ in L_p -space.

The characterization of

$$(5) \quad E_n^{(p)}(v_{\alpha, \beta}; f) = O(n^{-\gamma})$$

by the integral modulus of continuity of f is due to professor GÉZA FREUD [1]. In this paper of us, we give an alternative condition for (5) by the derivatives of the polynomials of best approximation. The analogue of our Theorem for trigonometric approximation of periodic function is due to M. ZAMANSKY [6], G. I. SUNOUCHI [5] and for algebraic approximation to S. PAWELKE [4].

Theorem. Let $v_{\alpha,\beta} f \in L_p(-\infty, \infty)$. Let r be a positive integer, and let γ be such positive real number, that $(1-1/\alpha)r-\gamma > 0$. For the polynomial $g_n(f; x) = g_n(L_p; f; x)$ the following statements are equivalent:

$$(6) \quad (i) \quad E_n^{(p)}(v_{\alpha,\beta}; f) = O(n^{-\gamma}).$$

$$(7) \quad (ii) \quad \|v_{\alpha,\beta} q_n^{(r)}(f)\|_p = O(n^{(1-1/\alpha)r-\gamma}).$$

PROOF. The proof will be based on the following inequalities:

1) For each real λ there is a constant M_λ with the property that for each sequence $0 < u_k \leq u_{k+1} \leq \dots \leq u_l$, such that $2 \leq u_i/u_{i-1} \leq 4$ for $k < i \leq l$, and for each positive decreasing function $\varphi(u)$ defined for $u \geq 0$,

$$(8) \quad \sum_{i=k}^l u_i^\lambda \varphi(u_i) \leq M_\lambda \sum_{n=[(1/2)u_k]}^{u_l-1} (n+1)^{\lambda-1} \varphi(n).$$

(See G.G. LORENTZ [4, p. 58—59])

2) By a not yet published result of Freud **) for each $p_n \in P_n$, we have

$$(9) \quad \|v_{\alpha,\beta} p_n^{(r)}\|_p \leq M_{p,r,\alpha,\beta}^{(1)} n^{(1-1/\alpha)r} \|v_{\alpha,\beta} p_n\|_p.$$

3) If f is an r -time iterated integral function of $f^{(r)}$, and $v_{\alpha,\beta} f^{(r)} \in L_p(-\infty, \infty)$ then

$$(10) \quad E_n^{(p)}(v_{\alpha,\beta}; f) \leq M_{p,r,\alpha,\beta}^{(2)} n^{(1/\alpha-1)r} \|v_{\alpha,\beta} f^{(r)}\|_p.$$

(See G. FREUD [1]).

a) (i) $\rightarrow$ (ii): For $n \geq 1$, let k be such integer, that $2^k \leq n < 2^{k+1}$. We have

$$(11) \quad \begin{aligned} q_n^{(r)}(x) &= [q_n(x) - q_{2^k}(x)]^{(r)} + [q_{2^k}(x) - q_{2^{k-1}}(x)]^{(r)} + \dots + \\ &+ [q_2(x) - q_1(x)]^{(r)} + [q_1(x) - q_0(x)]^{(r)}. \end{aligned}$$

For $E_n = E_n^{(p)}(v_{\alpha,\beta}; f)$, we notice that

$$(12) \quad \|v_{\alpha,\beta} (q_{2^i} - q_{2^{i-1}})\|_p \leq 2E_{2^{i-1}} \quad (i = 1, 2, \dots, k), \quad \|v_{\alpha,\beta} (q_n - q_{2^k})\|_p \leq 2E_{2^k}$$

Now, from (11), (9) and (12), using the inequality (8) we get

$$(13) \quad \begin{aligned} \|v_{\alpha,\beta} q_n^{(r)}\|_p &\leq M_{p,r,\alpha,\beta}^{(1)} \{n^{(1-1/\alpha)r} 2E_{2^k} + 2^{k(1-1/\alpha)r} 2E_{2^{k-1}} + \dots + 2^{(1-1/\alpha)r} 2E_1 + 2E_0\} \leq \\ &\leq 2^{(1-1/\alpha)r+1} M_{p,r,\alpha,\beta}^{(1)} \left[\sum_{i=0}^k 2^{i(1-1/\alpha)r} E_{2^i} + E_0 \right] \leq \\ &\leq 2^{(1-1/\alpha)r+2} M_{p,r,\alpha,\beta}^{(1)} M_{r,\alpha} \sum_{l=0}^{2^k-1} (l+1)^{(1-1/\alpha)r-1} E_l \leq \\ &\leq 2^{(1-1/\alpha)r+2} M_{p,r,\alpha,\beta}^{(1)} M_{r,\alpha} \sum_{l=0}^{n-1} (l+1)^{(1-1/\alpha)r-1} E_l. \end{aligned}$$

*) The $M_{\lambda,p,\dots}^{(k)}$ denote positive constants, which depend only on $\lambda, p, \dots$

**) Special cases are delt in [2]. The more general case was included in prof. Freud's lectures at the Szeged University in the year 1973/74.

Hence follows that, if $E_l = O(l^{-\gamma})$, then since $(1-1/\alpha)r-1-\gamma > -1$, we have

$$\|v_{\alpha, \beta} q_n^{(r)}\|_p = O(n^{(1-1/\alpha)r-\gamma}),$$

which was to be proved.

b) (ii) $\rightarrow$ (i): If $\|v_{\alpha, \beta} q_n^{(r)}(f)\|_p \leq An^{(1-1/\alpha)r-\gamma}$,

then from (10) we obtain

$$(14) \quad \|v_{\alpha, \beta} \{q_{2n}(f) - q_n[q_{2n}(f)]\}\|_p \leq M_{p, r, \alpha, \beta}^{(2)} \cdot A \cdot n^{-\gamma}.$$

Furthermore, since $g_n[g_{2n}(f)] \in P_n$, we have

$$(15) \quad \begin{aligned} \|v_{\alpha, \beta} \{q_{2n}(f) - q_n[q_{2n}(f)]\}\|_p &\geq \|v_{\alpha, \beta} \{f - q_n[q_{2n}(f)]\}\|_p - \\ &- \|v_{\alpha, \beta} [f - q_{2n}(f)]\|_p \geq E_n^{(p)}(v_{\alpha, \beta}; f) - E_{2n}^{(p)}(v_{\alpha, \beta}; f). \end{aligned}$$

From (3), (14) and (15) we get

$$E_{2^k} = \sum_{i=k}^{\infty} \{E_{2^i} - E_{2^{i+1}}\} \leq M_{p, r, \alpha, \beta}^{(2)} \cdot A \cdot \sum_{i=k}^{\infty} 2^{-i\gamma} = O(2^{-k\gamma}),$$

from which follows, that $E_n = O(n^{-\gamma})$. This completes our proof.

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