

Convergence of Hermite—Fejér interpolation polynomial on the extended nodes

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1. In 1969 BERMAN [1] proved a surprising result that the *Hermite—Fejér interpolation process* $H_n(f, x)$ of degree $\leq 2n-1$ defined by the conditions:

$$(1) \quad H_n(f, x_k) = f(x_k), \quad H'_n(f, x_k) = 0, \quad k = 1, 2, \dots, n$$

constructed on the nodes

$$(2) \quad x_0 = 1, \quad x_k = \cos \frac{2k-1}{2n} \pi, \quad k = 1, 2, 3, \dots, n$$

or

$$(3) \quad x_{n+1} = -1, \quad x_k = \cos \frac{2k-1}{2n} \pi, \quad k = 1, 2, \dots, n$$

diverges for a simple function $f(x) = |x|$ at $x=0$, while according to the well known result of FEJÉR [3] the polynomial $H_n(f, x)$ constructed on the nodes $x_k = \cos \frac{2k-1}{2n} \pi$, $k=1, 2, \dots, n$, the roots of the Tchebycheff polynomial of the first kind, $T_n(x)$, converges to a continuous function $f(x)$ in $[-1, 1]$. Recently we [6] proved the divergence of $H_n(f, x)$ for nodes (2) or (3) every where in $[-1, 1]$.

The point system (2) or (3) is the extension of the Tchebycheff nodes by $+1$ or -1 and the Tchebycheff nodes are best interpolation nodes. It appears due to the result of BERMAN [1] that the addition of $+1$ or -1 in any system of nodes disturbs the convergence of the Hermite—Fejér interpolation polynomial. In this paper we show that this is not always true. We prove

Theorem 1. *The Hermite—Fejér interpolation process $H_n(f, x)$ constructed on the points $\cos \frac{2k-1}{2n+1} \pi$ ($k=0, \dots, n+1$) converges uniformly to $f(x)$ in $[-1, 1]$, when $f(x)$ is continuous in $[-1, 1]$.*

Theorem 2. *The Hermite—Fejér interpolation process $H_n(f, x)$ constructed on the points $\cos \frac{2k}{2n+1} \pi$ ($k=0, \dots, n$) converges uniformly to $f(x)$ in $[-1, 1]$, when $f(x)$ is continuous in $[-1, 1]$.*

The proof of theorem 2 runs exactly on the same lines as proof of theorem 1. So we omit the proof.

2. For our purpose we consider the system of nodes

$$(4) \quad x_{n+1} = -1, \quad x_k = \cos \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, 3, \dots, n.$$

The points $\cos \frac{2k-1}{2n+1} \pi$ are the roots of Jacobi polynomial $P_n^{(-1/2, 1/2)}(x)$, which is identical with

$$W_n(x) = \text{const} \left[\frac{\cos (2n+1)\theta/2}{\cos \theta/2} \right]_{x=\cos \theta},$$

satisfying the differential equation

$$(5) \quad (1-x^2)W_n''(x) + (1-2x)W_n'(x) + n(n+1)W_n(x) = 0.$$

It is well known that

$$\frac{W_n'(x)}{W_n(x)} = \sum_{k=1}^n \frac{1}{x-x_k}$$

and

$$\frac{W_n'^2(x) - W_n''(x)W_n(x)}{W_n^2(x)} = \sum_{k=1}^n \frac{1}{(x-x_k)^2},$$

which give

$$(6) \quad \sum_{k=1}^n \frac{1}{(1+x_k)} = \frac{n(n+1)}{3}.$$

$$(7) \quad \sum_{k=1}^n \frac{1}{(1-x_k)} = n(n+1)$$

$$(8) \quad \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{1}{(x-x_k)} = \frac{\sin (2n+1)\theta}{(2n+1) \sin \theta} - \frac{\cos^2 (2n+1)\theta/2}{(2n+1)^2 \cos^2 \theta/2}$$

and

$$(9) \quad \begin{aligned} \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{1}{(x-x_k)^2} &= 1 - \frac{(2 \sin^2 \theta/2 - \cos^3 \theta/2) \sin (2n+1)\theta}{(2n+1) \sin \theta} + \\ &+ \frac{(2 \cos 3\theta/2 - \cos \theta) \cos^2 (2n+1)\theta/2}{(2n+1)^2 \cos^2 \theta/2}. \end{aligned}$$

3. Hermite—Fejér interpolation process $H_n(f, x)$ constructed on the nodes (4) is given by

$$(10) \quad \begin{aligned} H_n(f, x) &= [1 + 2/3n(n+1)(x+1)] \frac{W_n^2(x)}{W_n^2(-1)} f(-1) + \\ &+ 2 \sum_{k=1}^n f(x_k) \left[1 - \frac{1}{1-x_k^2} (x-x_k) \right] \frac{(1+x)^2 W_n^2(x)(1-x_k)}{(2n+1)^2 (x-x_k)^2}. \end{aligned}$$

To prove theorem 1, we need the following lemmas.

Lemma 1. For any $f \in C$ and $-1 < x < 1$

$$(11) \quad \lim_{n \rightarrow \infty} \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{x-x_k} = 0,$$

where

$$x_k = \cos \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, 3, \dots, n.$$

PROOF. First, we prove this lemma, when $f(x)$ is a polynomial. So it suffices to take $f(x) = x^j$, j finite integer ≥ 0 . For $j=0$ (11) follows at once from (8). For $j \geq 1$, we set

$$\begin{aligned} A_{jn} &= \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{x_k^j}{(x-x_k)} = \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{\{x-(x-x_k)\}^j}{(x-x_k)} = \\ &= \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \left\{ \sum_{k=1}^n \frac{x^j}{(x-x_k)} + \sum_{k=1}^n \sum_{i=1}^j jC_i(-1)^i(x-x_k)^i x^{j-i} \right\}. \end{aligned}$$

Since

$$\left| \sum_{k=1}^n \sum_{i=1}^j jC_i(-1)^i(x-x_k)^i x^{j-i} \right| \leq 2^j n(2^j - 1),$$

therefore,

$$\left| A_{jn} - \frac{2(1+x)W_n^2(x)x^j}{(2n+1)^2} \sum_{k=1}^n \frac{1}{x-x_k} \right| \leq \frac{2(1+x)W_n^2(x)2^j n(2^j - 1)}{(2n+1)^2}.$$

Hence

$$\lim_{n \rightarrow \infty} A_{jn} = 0,$$

owing to (8) and the inequality $(1+x)W_n^2(x) \leq 2$.

Now, let $f(x) \in C$. We know that for $\varepsilon > 0$, there exists a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon, \quad \text{for } -1 \leq x \leq 1.$$

Setting

$$R_n(f) = \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)},$$

we get

$$(12) \quad |R_n(f)| \leq |R_n(f-P)| + |R_n(P)|.$$

Since (11) holds, when $f(x)$ is a polynomial, therefore there exists a number N such that

$$(13) \quad |R_n(P)| < \varepsilon \quad \text{for all } n \geq N.$$

Further

$$\begin{aligned}
 (14) \quad |R_n(f-P)| &\equiv \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{|f(x_k)-P(x_k)|}{|x-x_k|} \equiv \\
 &\equiv \frac{2\varepsilon(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{|x-x_k|}{(x-x_k)^2} \equiv \frac{4\varepsilon(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{1}{(x-x_k)^2} = \\
 &= 4\varepsilon \left\{ \frac{1}{\sin^2 \theta} - \frac{(2 \sin^2 \theta/2 - \cos^3 \theta/2) \sin (2n+1) \theta}{(2n+1) \sin^3 \theta} + \frac{2(\cos^3 \theta/2 - \cos \theta) \cos^2 (2n+1) \theta/2}{(2n+1)^2 \sin^2 \theta \cos^2 \theta/2} \right\}, \\
 &\quad -1 < x < 1.
 \end{aligned}$$

Hence from (13), (14) and (12) it follows that

$$\lim_{n \rightarrow \infty} R_n(f) = 0.$$

Thus lemma 1 is proved.

Lemma 2. For any $f \in C$ and $-1 < x < 1$

$$(15) \quad \lim_{n \rightarrow \infty} \frac{2(1+x)(1-x^2)W_n(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)^2} = f(x),$$

where

$$x_k = \cos \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, \dots, n.$$

PROOF. As in the proof of lemma 1, it is sufficient to prove (15) only for $f(x)=x^j$, $j=0, 1, 2, \dots$. For $f(x)=1$, (15) follows atonce from (9). Further let $f(x)=x$, then

$$\begin{aligned}
 B_{1n} &= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{x_k}{(x-x_k)^2} = \\
 &= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \left\{ \sum_{k=1}^n \frac{x}{(x-x_k)^2} - \sum_{k=1}^n \frac{1}{x-x_k} \right\}.
 \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} B_{1n} = x$$

owing to (8) and (9).

Now, let $f(x)=x^j$, $j \geq 2$. Then

$$\begin{aligned}
 B_{jn} &= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{x_k^j}{(x-x_k)^2} = \\
 &= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{\{x-(x-x_k)\}^j}{(x-x_k)^2} = \\
 &= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \left\{ \sum_{k=1}^n \frac{x^j}{(x-x_k)^2} - \sum_{k=1}^n \frac{jx^{j-1}}{(x-x_k)} + \right. \\
 &\quad \left. + \sum_{k=1}^n \sum_{i=2}^j jC_i(-1)^i x^{j-i} (x-x_k)^{i-2} \right\},
 \end{aligned}$$

which gives

$$\begin{aligned} & \left| B_{jn} - \frac{2(1+x)(1-x^2)W_n^2(x)x^j}{(2n+1)^2} \sum_{k=1}^n \frac{1}{(x-x_k)^2} + \right. \\ & \quad \left. + \frac{2(1+x)(1-x^2)W_n^2(x)jx^{j-1}}{(2n+1)^2} \sum_{k=1}^n \frac{1}{x-x_k} \right| \leq \\ & \leq \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \left| \sum_{k=1}^n \sum_{i=2}^j jC_i(-1)^i x^{j-i} (x-x_k)^{i-2} \right|. \end{aligned}$$

But

$$\left| \sum_{k=1}^n \sum_{i=2}^j jC_i(-1)^i x^{j-i} (x-x_k)^{i-2} \right| < 2^{j-2}n(2^j-1-j),$$

therefore,

$$\begin{aligned} & \left| B_{jn} - \frac{2x^j(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{1}{(x-x_k)^2} + \right. \\ & \quad \left. + \frac{2jx^{j-1}(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{1}{x-x_k} \right| < \frac{2^{j-1}n(2^j-1-j)(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}. \end{aligned}$$

Taking limit, we get

$$\lim_{n \rightarrow \infty} B_{jn} = x^j$$

owing to (8), (9) and the inequality $(1+x)W_n^2(x) \leq 2$.

Thus we have proved that (15) is true when $f(x)$ is a polynomial. Further by repeating the arguments as in the proof of lemma 1, we can prove that (15) is true for arbitrary $f(x) \in C[-1, 1]$.

Similarly we can prove

Lemma 3. For any $f \in C$ and $x_k = \cos \frac{2k-1}{2n+1} \pi$, $k = 1, 2, \dots, n$

$$(16) \quad \lim_{n \rightarrow \infty} \frac{1}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k} = \frac{f(-1)}{12}$$

and

$$(17) \quad \lim_{n \rightarrow \infty} \frac{1}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1-x_k} = \frac{f(1)}{4}.$$

PROOF of theorem 1. From (10), we have

$$\begin{aligned} (18) \quad H_n(f, x) &= \left[1 + \frac{2}{3} n(n+1)(x+1) \right] \frac{W_n^2(x)}{W_n^2(-1)} f(-1) + \\ &+ \frac{2(1+x)^2 W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n f(x_k) \left\{ \frac{1-x_k}{(x-x_k)^2} - \frac{1}{(x-x_k)(1+x_k)} \right\}. \end{aligned}$$

Since

$$(1+x) \left[\frac{1-x_k}{(x-x_k)^2} - \frac{1}{(x-x_k)(1+x_k)} \right] = \frac{(1-x^2)}{(x-x_k)^2} + \frac{x}{x-x_k} - \frac{1}{1+x_k},$$

therefore,

$$(19) \quad H_n(f, x) = \left[1 + \frac{2}{3} n(n+1)(x+1)\right] \frac{W_n^2(x)}{(2n+1)^2} f(-1) - \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k} + \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \times \left\{ (1-x^2) \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)^2} + x \sum_{k=1}^n \frac{f(x_k)}{x-x_k} \right\}.$$

Now taking limit, we get

$$(20) \quad \begin{aligned} \lim_{n \rightarrow \infty} H_n(f, x) &= \lim_{n \rightarrow \infty} \left[1 + \frac{2}{3} n(n+1)(x+1)\right] \frac{W_n^2(x)}{(2n+1)^2} f(-1) - \\ &- \lim_{n \rightarrow \infty} \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k} + \lim_{n \rightarrow \infty} \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)^2} + \\ &+ \lim_{n \rightarrow \infty} \frac{2x(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{x-x_k} = \\ &= \lim_{n \rightarrow \infty} (1+x)W_n^2(x) \left[\frac{f(-1)}{6} - \lim_{n \rightarrow \infty} \frac{2}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k} \right] + \\ &+ \lim_{n \rightarrow \infty} \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)^2} + 2x \lim_{n \rightarrow \infty} \frac{(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{x-x_k}. \end{aligned}$$

Applying lemmas 1 – 3 to (20), we get

$$\lim_{n \rightarrow \infty} H_n(f, x) = f(x) \quad \text{for } -1 < x < 1$$

owing to the inequality $(1+x)W_n^2(x) \leq 2$. Again putting $x=1$ in (10), we have

$$(21) \quad H_n(f, 1) = \left[1 + \frac{4}{3} n(n+1)\right] \frac{f(-1)}{(2n+1)^2} + \frac{4}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1-x_k} - \frac{4}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k}.$$

Taking limit, we get

$$\lim_{n \rightarrow \infty} H_n(f, 1) = \frac{f(-1)}{3} + \lim_{n \rightarrow \infty} \frac{4}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1-x_k} - \lim_{n \rightarrow \infty} \frac{4}{(2n+1)^2} \sum_{k=1}^n \frac{f(x_k)}{1+x_k},$$

which with the help of lemma 3 gives

$$\lim_{n \rightarrow \infty} H_n(f, 1) = f(1).$$

This proves theorem 1 for $x=1$. For $x=-1$ theorem 1 follows from continuity.

4. Quasi-Hermite-Fejér interpolation process $Q_n(f, x)$ defined by the conditions:

$$\begin{aligned} Q_n(f, 1) &= f(1), \quad Q_n(f, -1) = f(-1), \\ Q_n(f, x_k) &= f(x_k), \quad Q'_n(f, x_k) = 0, \quad k = 1, 2, \dots \end{aligned}$$

is given by

$$(22) \quad Q_n(f, x) = f(1) \frac{1+x}{2} \frac{W_n^2(x)}{W_n^2(1)} + f(-1) \frac{1-x}{2} \frac{W_n^2(x)}{W_n^2(-1)} + \\ + \sum_{k=1}^n f(x_k) \frac{1-x^2}{1-x_k^2} \left[1 + \left\{ \frac{2x_k}{1-x_k^2} - \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] \left[\frac{W_n(x)}{W_n'(x_k)(x-x_k)} \right]^2.$$

One can easily prove the following.

Theorem 3. *The quasi-Hermite-Fejér interpolation polynomial $Q_n(f, x)$ given by (22) constructed on the zeros of Jacobi polynomial $P_n^{(-1/2, 1/2)}(x)$ or $P_n^{(1/2, -1/2)}(x)$ converges uniformly to $f(x)$, when $f(x)$ is continuous in $[-1, 1]$.*

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