

On products of full linear rings

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To the memory of Andor Kertész

Recently GOLDMAN [2] showed that a ring (with unity element) is a product of full linear rings (of left vector spaces over division rings) if and only if R is Hausdorff and complete in the right *intrinsic* topology, which has a fundamental system of open neighborhoods of zero consisting of all right ideals K such that R/K is a completely reducible projective right R -module.

Ring theoretic characterizations of products of full linear rings were given by CHASE and FAITH [1]. It is the purpose of this note to show that both characterizations follow easily from Jacobson's density theorem as formulated in [6] and some easy results on bicommutators proved here. It is also shown that the intrinsic topology on R coincides with the I -adic topology. In general, the I -adic topology on a right R -module A has a fundamental system of open neighborhoods of zero consisting of all kernels of homomorphisms $A \rightarrow I^n$, where n is any natural number.

Given right R -modules I and A , we adopt the notation from [6]:

$$S_I(A) = \text{Hom}_{\text{End}(I)}(\text{Hom}_R(A, I), I)$$

where I is regarded as a left module over its ring of endomorphisms $\text{End}(I)$. In particular,

$$S_I(R) \cong \text{Hom}_{\text{End}(I)}(I, I) = \text{Bic}(I)$$

is the *bicommutator* of I , which is regarded as a right $S_I(R)$ -module. We recall the following from [6, Remark 5, 3]:

Lemma 0. *If I is a completely reducible right R -module, the canonical module homomorphism $A \rightarrow S_I(A)$ is the Hausdorff completion of A in the I -adic topology. In particular, the canonical ring homomorphism $R \rightarrow \text{Bic}(I)$ is the Hausdorff completion of R .*

Given a right R -module V and a set X , we write

$$XV = \sum_{\alpha \in X}^{\oplus} V_{\alpha}.$$

where $V_{\alpha} = V$ for all $\alpha \in X$. We denote by $p_{\alpha}: XV \rightarrow V$ and $k_{\alpha}: V \rightarrow XV$ the canonical projection and injection corresponding to $\alpha \in X$.

Lemma 1. *Let V and A be right R -modules, X any set. If X is finite or A is finitely generated, then*

$$S_{XV}(A) \cong S_V(A).$$

In particular,

$$\text{Bic}(XV) \cong \text{Bic}(V).$$

PROOF. Let $\beta \in X$ be fixed. For any $s \in S_{XV}(A)$ any $g \in \text{Hom}_R(A, V)$ we write

$$(g)s^* = p_\beta((k_\beta g)s).$$

(According to the usual convention, homomorphisms of left modules act from the right.) Then $*$ is an R -homomorphism $S_{XV}(A) \rightarrow S_V(A)$.

For any $t \in S_V(A)$ and $f \in \text{Hom}_R(A, XV)$, we write

$$t^+(f) = \sum_{\alpha \in X} k_\alpha((p_\alpha f)t).$$

For this formula to make sense, we require that $p_\alpha f = 0$ for all but a finite number of α . This will be so if X is finite or if A is finitely generated. A straight forward calculation shows that $+$ is the inverse of $*$.

Lemma 2. *Let A and I be right R -modules and $I = \sum_{\alpha \in X}^{\oplus} I_\alpha$, where $\text{Hom}_R(I_\alpha, I_\beta) = 0$ if $\alpha \neq \beta$. If X is finite or A is finitely generated, then*

$$S_I(A) \cong \prod_{\alpha \in X} S_{I_\alpha}(A).$$

In particular,

$$\text{Bic}(I) \cong \prod_{\alpha \in X} \text{Bic}(I_\alpha).$$

PROOF. We denote by $p_\alpha: I \rightarrow I_\alpha$ and $k_\alpha: I_\alpha \rightarrow I$ the canonical projection and injection corresponding to $\alpha \in X$.

To each $s \in S_I(A)$ we assign $s^* \in \prod_{\alpha \in X} S_{I_\alpha}(A)$ by defining, for any $f_\alpha \in \text{Hom}_R(A, I_\alpha)$, the component $s^* \in S_{I_\alpha}(A)$ by

$$(f_\alpha)s_\alpha^* = p_\alpha((k_\alpha f_\alpha)s).$$

To each $t \in \sum_{\alpha \in X} S_{I_\alpha}(A)$, with components $t_\alpha \in S_{I_\alpha}(A)$, we assign $t^+ \in S_I(A)$ by defining, for any $f \in \text{Hom}_R(A, I)$,

$$(f)t^+ = \sum_{\alpha \in X} k_\alpha((p_\alpha f)t_\alpha).$$

This formula makes sense if $p_\alpha f = 0$ for all but a finite number of $\alpha \in X$, and this is so if X is finite or A is finitely generated.

Under the assumption that $\text{Hom}_R(I_\alpha, I_\beta) = 0$ for $\alpha \neq \beta$, one easily computes that $+$ is the inverse of $*$.

We recall that the left socle and right socle of a semiprime ring coincide [5, § 3.4, Proposition 4].

Lemma 3. *Let R be a semiprime ring and suppose that its socle S is essential as a left ideal. Then S is also essential as a right ideal, and the bicommutator of the right R -module S_R is an essential extension of the left R -module ${}_R R$.*

PROOF. Suppose $0 \neq r \in R$. Since ${}_R S$ is essential, Rr contains a nonzero idempotent $e = r'r$. Then $r'r'er'r = e^3 = e \neq 0$, hence $0 \neq rer' \in S$. Therefore, S_R is essential.

Since R is semiprime, the right annihilator of S is zero. It follows that the canonical mapping $R \rightarrow \text{Bic}(S_R)$ is a monomorphism. Let $0 \neq q \in \text{Bic}(S_R)$, then we can find an idempotent e such that $0 \neq eq \in S \subseteq R$. Therefore, $\text{Bic}(S_R)$ is an essential extension of ${}_R R$.

Lemma 4. *For a semiprime ring with socle S , the S -adic topology coincides with the intrinsic topology.*

PROOF. Let K be a fundamental open neighborhood of zero in the S -adic topology of R_R , that is, the kernel of a homomorphism $f: R \rightarrow S^n$. Then $R/K \cong \text{im } f \subseteq S^n \subseteq S$ is a direct sum of minimal right ideals of the form eR , where $e^2 = e$, hence completely reducible and projective. Therefore, K is a fundamental open neighborhood of zero in the intrinsic topology.

Conversely, suppose R/K is completely reducible and projective. Then $K = eR$, for some idempotent $e \in R$, and $R/K \cong (1-e)R \subseteq S$. Therefore, K is the kernel of a homomorphism $R \rightarrow S$.

For completeness, we shall also record the following wellknown facts.

Lemma 5. *Let ${}_D V$ be a left vector space over a division ring D and $R = \text{Hom}_D(V, V)$. Then R is von Neumann regular, self-injective and its socle is essential as a left ideal.*

PROOF. It is well-known that R is regular [see e.g. 5, § 4.4, Proposition 1]. Thus V_R is flat, and since ${}_D V$ is injective, it follows that ${}_R R$ is injective [5, § 5.3, Proposition 3]. To see that the socle S is essential as a left ideal, let $0 \neq r \in R$, then $vr \neq 0$ for some $v \in V$. Let $V = Dv \oplus W$ and define $e \in R$ by $ve = 1$, $We = 0$. Then $ver = vr \neq 0$ hence $er \neq 0$. But $er \in Rer \subseteq S$, since Re is a minimal left ideal of R , as is easily checked.

Theorem. *The following properties of a ring R are equivalent:*

- (1) R is Hausdorff and complete in the right intrinsic topology.
- (2) R is Hausdorff and complete in the S -adic topology, where S is the socle of R regarded as a right R -module.
- (3) R is Hausdorff and complete in the I -adic topology, where I is some completely reducible right R -module.
- (4) R is isomorphic to the bicommutator of some completely reducible right R -module.
- (5) R is a direct product of full linear rings of left vector spaces.
- (6) R is von Neumann regular, left self-injective and its socle is essential as a left ideal.
- (7) R is semiprime left self-injective and its socle is essential as a left ideal.
- (8) The canonical homomorphism $R \rightarrow \text{Bic}(S)$, where S is the right socle of R , is an isomorphism.

PROOF. We shall establish the following implications:

(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2).

(1) $\Leftrightarrow$ (2) by Lemma 4, because R is semiprime under either hypothesis.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) follows from Lemma 0.

(4) $\Rightarrow$ (5): Suppose R is isomorphic to the bicommutator of a completely reducible module I . We may write $I = \sum_{\alpha \in X}^{\oplus} I_{\alpha}$, where the I_{α} are the homogeneous components of I , and $I_{\alpha} = X_{\alpha} V_{\alpha}$, X_{α} being a set and V_{α} a fixed minimal submodule of I_{α} . By lemmas (2) and (1), we have

$$R \cong \text{Bic}(I) \cong \prod_{\alpha \in X} \text{Bic}(I_{\alpha}) \cong \prod_{\alpha \in X} \text{Bic}(V_{\alpha}) = \prod \text{End}_{D_{\alpha}}(V_{\alpha}),$$

where $D_{\alpha} = \text{End}_R(I_{\alpha})$.

(5) $\Rightarrow$ (6): Each $R_{\alpha} = \text{End}_{D_{\alpha}}(V_{\alpha})$ is regular, self-injective and its socle is essential as a left ideal, by Lemma 5. Therefore, R is left self-injective [5, § 4.3, Proposition 9] and von Neumann regular (obvious), and it is easily seen that the socle of R is left essential.

(6) $\Rightarrow$ (7) is trivial.

(7) $\Rightarrow$ (8): By Lemma 3, the bicommutator of S_R is an essential extension of ${}_R R$. Since ${}_R R$ is injective, $R \cong \text{Bic}(S_R)$ canonically.

(8) $\Rightarrow$ (2) follows from Lemma 0.

COROLLARY 1. *The complete ring of left quotients Q of a ring R is a product of full linear rings of left vector spaces if and only if the following conditions hold:*

(1) *The left singular ideal Z of R is zero.*

(2) *Every nonzero left ideal of R contains a nonzero uniform left ideal.*

PROOF. Assume the conditions. By (1), Q is von Neumann regular and left self-injective [5, § 4.5]. It will follow from the theorem that Q is a full linear ring if we show that its socle is left essential. So, let L be a nonzero left ideal of Q , we want to show that L contains a minimal left ideal. Now $L \cap R \neq 0$; hence, by (2), it contains a uniform left ideal $U \neq 0$ of R , and $QU \subseteq L$. We claim that QU is a minimal left ideal of Q . Since Q is regular, it suffices to check that QU is uniform. (Let $0 \neq a, b \in QU$. Since Q is a left ring of quotients of R , we can find $r, s \in R$ such that $0 \neq ra, sb \in U$. Therefore $Rra \cap Rrb \neq 0$, hence $Qa \cap Qb \neq 0$. Thus QU is uniform.)

Conversely, suppose Q is a product of full linear rings. Then, by the theorem, Q is von Neumann regular, left self-injective, and its socle S is left essential. By [5, § 4.5], $Z = 0$. Let L be a nonzero left ideal of R , then $QL \cap S \neq 0$, and so QL contains a minimal left ideal V of Q . Since Q is a ring of left quotients of R , $R \cap V$ is a nonzero uniform left ideal of R . (For, if $0 \neq a, b \in R \cap V$, we can find $q, q' \in Q$ such that $0 \neq qa = q'b \in V$, then find $r \in R$ such that $rq \in R$, $rq' \in R$ and $0 \neq rqa = rq'b \in V$.) Let $0 \neq v \in V \subseteq QL$, we can find $r \in R$ such that $0 \neq rv \in L$. Hence Rrv will be a nonzero left ideal contained in $V \cap L$, hence a uniform left ideal contained in L .

The sufficiency of conditions (1) and (2) is due to JOHNSON [4, Theorem 3.1], see also [3, Theorem 3.8].

COROLLARY 2. *Let R be a semiprime ring whose socle is an essential left ideal. Then its complete ring of left quotients is a product of full linear rings of left vector spaces.*

PROOF. In view of Corollary 1, we need only verify the two conditions.

(1) Since every essential left ideal contains the socle S , $(S \cap Z)^2 \subseteq SZ = 0$, hence $S \cap Z = 0$, and so $Z = 0$.

(2) Every nonzero left ideal has nonzero intersection with S , hence contains a minimal left ideal.

This corollary is a generalization of a theorem by UTUMI [5, § 4.3, Proposition 7] also found in [1]. One could use it for obtaining yet another proof of Goldie's theorem on semiprime rings.

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