

Subproducts and subdirect products

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To the memory of Prof. A. Kertész

§ 1. Introduction

In the following R is a commutative ring ($1 \in R$), $\{M_i\}_{i \in I}$ a (non-empty) set of R -modules $M_i \neq 0$, F an R -module, and $\alpha_i: M_i \rightarrow F (i \in I)$ a set of R -epimorphism. We consider the set $M \subset \prod_{i \in I} M_i$ of those elements $m = (m_i)_{i \in I} \in \prod_{i \in I} M_i$ such that the following relations are satisfied:

$$(1) \quad \sum_{i \in I} r_{ji} \alpha_i(m_i) = 0 \quad (j \in J),$$

where I and J are index sets, and where every sum in (1) contains only a finite number of terms $\neq 0$. M is a submodule of $\prod_{i \in I} M_i$, called a *subproduct* M of the M_i , denoted by

$$(2) \quad M = \{M_i; \alpha_i; F; \sum_{i \in I} r_{ji} \alpha_i(m_i) = 0; j \in J\}.$$

A well known example of (2) is the *special subdirect product*

$$M = \bigotimes_{i \in I} M_i(\alpha_i; F),$$

with relations

$$(3) \quad \alpha_i(m_i) - \alpha_k(m_k) = 0 (\forall i, k \in I).$$

The relations (1) are related with a system of linear equations

$$(4) \quad \sum_{i \in I} r_{ji} x_i = 0, \quad j \in J,$$

where $x_i = \alpha_i(m_i)$, $i \in I$. This system is a system of *homogeneous linear equations* over F , so we have always the zero solution of (4). If $K_i = \text{Ker}(\alpha_i)$, $i \in I$, then M contains the direct product $K = \prod_{i \in I} K_i$ as a submodule. If therefore the number of $K_i \neq 0$ is infinite, M cannot be the direct sum of the M_i .

Therefore that M is the direct sum $\bigoplus_{i \in I} M_i$, it is necessary that $r_{ji}\alpha_i(m_i)=0$ for all i, j , and all m_i . With this condition only, M is still $\prod_{i \in I} M_i$, so I must be finite; according to our assumptions about the terms $r_{ji}\alpha_i(m_i)$ in (1), we can in this case omit the relations (1). Concerning the solutions of the equations (4): If $X = \langle \dots, x_i, \dots \rangle_{i \in I}$, $g_j = \sum_{i \in I} r_{ji}x_i$, and $Y = \langle \dots, g_j, \dots \rangle_{j \in J}$, then the solutions of (4) correspond in a one to one way with the elements of the R -module $\text{Hom}_R(X/Y, F)$. Indeed, if $(\dots, f_i, \dots)$ satisfies (4), then there is an R -homomorphism $\varphi: X/Y \rightarrow F$, determined by $\varphi(\bar{x}_i) = \varphi(x_i + Y) = f_i (i \in I)$. Conversely, if $\psi \in \text{Hom}_R(X/Y, F)$, such that $\psi(\bar{x}_i) = \psi(x_i + Y) = f_i (i \in I)$, then $x_i = f_i (i \in I)$ is a solution of (4). If $\varphi \in \text{Hom}_R(X/Y, F)$ determines the solution $x_i = f_i (i \in I)$ of (4), then $r\varphi$ determines the solution $rx_i = rf_i (i \in I)$.

A subproduct has the following *universal* property:

Theorem 1.1. *Let the subproduct M be defined by (2) and let π_i be the canonical projection $\pi_i: M \rightarrow M_i (i \in I)$, then $\sum_{i \in I} r_{ji}\alpha_i\pi_i = 0 (\forall j \in J)$. Conversely, if M' is an R -module, $\varrho_i: M' \rightarrow M_i (\forall i \in I)$ an R -homomorphism, with $\sum_{i \in I} r_{ji}\alpha_i\varrho_i = 0 (\forall j \in J)$, then there exists a unique $\vartheta: M' \rightarrow M$, satisfying $\pi_i\vartheta = \varrho_i (\forall i \in I)$.*

$$\begin{array}{ccccc}
 M & \xrightarrow{\pi_i} & M_i & \xrightarrow{\alpha_i} & F \\
 & \searrow \phi & \nearrow \varrho_i & & \\
 & M' & & &
 \end{array}$$

PROOF. If $\varrho_i(m') = m_i (i \in I)$, $\sum_{i \in I} r_{ji}\alpha_i(m_i) = 0$, then the element $m = (m_i)_{i \in I} \in M$ and we define ϑ by $\vartheta(m') = m$; then $\pi_i\vartheta = \varrho_i$. If $\pi_i\vartheta = \pi_i\vartheta'$, then $\pi_i(\vartheta - \vartheta')m' = 0 (\forall i \in I)$, or $\vartheta(m') = \vartheta'(m')$. The solutions of the equations (4) form an R -module

$$S = \{(\dots, f_i, \dots) \mid \sum_{i \in I} r_{ji}f_i = 0; j \in J\};$$

with (f_i) and (f'_i) , we know that $(f_i + f'_i)$ and (rf_i) are also solutions of (4). For fixed $i \in I$ the components f_i of the solutions form a submodule F_i of F . Therefore S is a subdirect product

$$S = \bigtimes_{i \in I} F_i,$$

uniquely determined by the subproduct (2).

If $N_i = \alpha_i^{-1}F_i (i \in I)$, then we see that the subproduct M is a subdirect product of the R -modules $N_i (i \in I)$:

$$(5) \quad M = \bigtimes_{i \in I} N_i(\alpha_i; F_i; \sum_{i \in I} r_{ji}\alpha_i(n_i) = 0; j \in J).$$

If we have a set of subproducts $M^{(j)}$ with the same set $\{M_i; \alpha_i; F\}$, but evt. with different relation systems, then it is clear that the systems of relations *together* define a subproduct

$$M = \bigcap_j M^{(j)} \subset \prod_{i \in I} M_i.$$

This means that the subproduct, defined by (2) can be considered as the *intersection* of the *one-relation* subproducts

$$(6) \quad M^{(j)} = \{M_i; \alpha_i; F; \sum_i r_{ji} \alpha_i(m_i) = 0, \text{ fixed } j \in J\}, \quad M = \bigcap_j M^{(j)}.$$

The solutions of (4) correspond in a one to one way with the elements of $\text{Hom}_R(X/Y, F)$; now $M = \bigcap_j M^{(j)}$, where $M^{(j)}$ is defined by only one relation $\sum_i r_{ji} \alpha_i(m_i) = 0$ (j fixed).

The elements φ_j of $\text{Hom}_R(X/Y_j, F)$, where $Y_j = \langle g_j \rangle$, correspond in a one to one way with the solutions $(\dots, f_i^{(j)}, \dots)$ of $g_j = 0$ by $\varphi_j(\bar{x}_i) = f_i^{(j)}$. It is clear that the set $(\dots, \varphi_j, \dots)_{j \in J}$ defines a solution of (4) (i.e. defines an element of $\text{Hom}_R(X/Y, F)$) if for any two indices $j_1, j_2 \in J$ we have

$$\varphi_{j_1}(\dots, \bar{x}_i, \dots) = \varphi_{j_2}(\dots, \bar{x}_i, \dots).$$

If we denote by $H = \text{Hom}_R(X/Y, F)$ and by $H^{(j)} = \text{Hom}_R(X/Y_j, F)$, $j \in J$, then H is a special subdirect product of the $H^{(j)}$ ($j \in J$), determined by the homomorphisms

$$\beta^{(j)} : \varphi_j \mapsto (\dots, f_i^{(j)}, \dots, f_k^{(j)}, \dots).$$

§ 2. Special cases of subproducts

A. Let us consider the subproduct (2) of § 1 as the intersection of the one-relation subproducts $M^{(j)}$ (see § 1, (6)). Let

$$(7) \quad r_{ji_1} \alpha_{i_1}(m_{i_1}) + \dots + r_{ji_k} \alpha_{i_k}(m_{i_k}) = 0$$

be the corresponding relation of $M^{(j)}$, with corresponding equation

$$r_{ji_1} x_1 + \dots + r_{ji_k} x_k = 0.$$

Then we have

Theorem 2.1. $M^{(j)}$ is a subdirect product of the M_i ($i \in I$), exactly if $r_{ji_0} F \subset \sum_{i \neq i_0} r_{ji} F$ ($\forall i_0; r_{j_0} \neq 0$). Indeed, this follows from the fact that for any $f \in F$ the element $r_{ji_0} f$ can be written as $\sum_{i \neq i_0} r_{ji} f_i$ for suitable $f_i \in F$.

Corollary 2.2. Let the subproduct M be given by (2) with coefficients $r_{ji} \neq 0$, then M can be considered as the intersection of R -modules $M^{(j)}$ ($j \in J$) — where each $M^{(j)}$ is a subdirect product of the M_i — if and only if $r_{ji_0} F \subset \sum_{i \neq i_0} r_{ji} F$ ($\forall i_0 \in I, \forall j \in J$).

B. In the special case that we have two R -modules M_1, M_2 , $\alpha_i : M_i \rightarrow F$ ($i = 1, 2$), the two R -epimorphisms and the *only* relation $r_1 \alpha_1(m_1) + r_2 \alpha_2(m_2) = 0$, together with the conditions that $r_1 F = r_2 F$, then M is a subdirect product of M_1 and M_2 .

But this subdirect product must be a special subdirect product of M_1 and M_2 . If we define

$$N_1 = \{m_1 \in M_1 | (m_1, 0) \in M\}, \quad N_2 = \{m_2 \in M_2 | (0, m_2) \in M\},$$

then $N_1 = \{m_1 \in M_1 | r_1 \alpha_1(m_1) = 0\}$, etc.

This implies that there is a module $F_1 \subset F$ consisting of those $f_1 \in F$ with $r_1 f_1 = 0$. In the same way there exists a submodule $F_2 \subset F$ with $F_2 = \{f_2 \in F | r_2 f_2 = 0\}$ and

$$\alpha_1 N_1 = F_1, \quad \alpha_2 N_2 = F_2.$$

We prove that

$$M_1/N_1 \cong M_2/N_2.$$

Indeed, define a map $\varphi: M_1/N_1 \rightarrow M_2/N_2$ in the following way: if $m = (m_1, m_2) \in M$, then define

$$\varphi: m_1 + N_1 \mapsto m_2 + N_2;$$

if $m' = (m_1, m_2') \in M$, then $m_2 - m_2' \in N_2$ and $m_2 + N_2 = m_2' + N_2$.

Since M is a subdirect product of M_1 and M_2 , every m_2 occurs, so φ is „onto”. $\text{Ker}(\varphi) = \{(m_1, m_2) \in M | m_2 \in N_2\}$. Then it follows that $m_1 \in N_1$, hence $\text{Ker}(\varphi)$ is the zero element of M_1/N_1 ; this proves that $M_1/N_1 \cong M_2/N_2$. Since

$$M_1/N_1 \cong M_1/K_1/N_1/K_1 \cong F/F_1, \quad \text{and} \quad M_2/N_2 \cong F/F_2,$$

we have

$$F/F_1 \cong F/F_2.$$

Moreover M consists of all pairs (m_1, m_2) for which the natural homomorphisms $M_1 \rightarrow M_1/N_1$, $M_2 \rightarrow M_2/N_2$ map m_1 and m_2 upon corresponding cosets.

C. Theorem 2.3. Suppose that we have represented the subproduct M (of § 1, (2)) in the form (5) as a subdirect product of the N_i ; then the following statement holds: if the $N_i (i \in I)$ are injective and F is torsionfree, then M is an injective R -module.

PROOF. Let $L (\neq 0)$ be an ideal of the ring R , and $\varphi: L \rightarrow M$, $\pi_i: M \rightarrow N_i (i \in I)$ R -homomorphisms, where π_i is the canonical projection, then there exists a $\varphi_i: R \rightarrow N_i (i \in I)$, such that $\varphi_i \tau = \pi_i \varphi$. Suppose $\varphi_i(1) = n_i$, then for $\lambda \in L$, we have $\varphi_i(\lambda) = \lambda n_i$. If $\varphi(\lambda) = m \in M$, then $\varphi(\lambda) = (\dots, \lambda n_i, \dots, \lambda n_k, \dots)$, and this element belongs to $M = \bigtimes_{i \in I} N_i$, such that the images $\alpha(\lambda n_i)$ satisfy the relation

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\tau} & R \\ & & \downarrow \varphi & & \nearrow \varphi_i \\ & & M & & \\ & & \downarrow \pi_i & & \\ & & N_i & & \end{array}$$

$$\sum_{i \in I} r_{ji} \alpha_i(\lambda n_i) = 0 \quad (j \in J),$$

hence

$$\lambda\left(\sum_{i \in I} r_{ji} \alpha_i(n_i)\right) = 0 \quad (\forall \lambda \in L; j \in J).$$

Since F is torsionfree, $L \neq 0$, we find

$$\sum_{i \in I} r_{ji} \alpha_i(n_i) = 0 \quad (j \in J),$$

and that implies that the element $m_0 = (\dots, n_i, \dots) \in M$, and moreover, that $\varphi(\lambda) = \lambda m_0$ proving that M is injective.

Remark 1. If — in particular — $N_i = M_i$, and therefore $F_i = F (\forall i \in I)$, then the injectivity of the M_i and the torsionfreeness of F implies the injectivity of M .

Remark 2. In the same way one proves:

Corollary. If M_i is injective ($\forall i \in I$) and F is torsionfree, then M is injective.

D. The case of a subdirect product.

In general, the subproduct M (see § 1, (2)) is not a subdirect product of the $M_i (i \in I)$. From § 1 we know that the solutions of the corresponding equations (4) over the module F form an R -module $S = \{(\dots, f_i, \dots) \mid \sum_{i \in I} r_{ji} f_i = 0; j \in J\}$, and moreover $S = \bigtimes_{i \in I} F_i$. A necessary and sufficient condition therefore that M is a subdirect product of the M_i is, that $F_i = F (\forall i \in I)$. One can express this (necessary and sufficient) condition in another way:

(i) M is a subdirect product of the M_i , if and only if — for every $k \in I$ and for every $f \in F$ — the equations

$$(9) \quad \sum_{i \neq k} r_{ji} x_i = -r_{jk} f \quad (j \in J)$$

are solvable in F . Equivalent with this condition (i) is the condition

(ii) For every $k \in I$ and for every $f \in F$, there exists an element $\varphi \in \text{Hom}_R(X/Y, F)$ such that

$$\varphi(\bar{x}_k) = \varphi(x_k + Y) = f.$$

From the theory of linear equations over an R -module we know that a necessary condition for the solvability of the system (9) is the compatibility of the system (9) for every $k \in I$ and every $f \in F$.

Theorem 2.4. If the systems (9) are compatible for every $k \in I$ and every $f \in F$, and F is injective, then M is a subdirect product of the M_i .

§ 3. Other subproducts

1. Which subproducts (2) describe the cartesian product $M = \prod_{i \in I} M_i$? In that case all elements $m = (m_i)_{i \in I} \in \prod_{i \in I} M_i$ have to satisfy the relations (1). Taking for $m = (\dots, 0, m_i, 0, \dots)$ we find as necessary conditions

$$(10) \quad r_{ji} \alpha_i(m_i) = 0$$

and we have to take care of the fact that $\alpha_i(m_i)$ runs through all elements of F , i.e. $r_{ji}f=0(\forall i \in I, j \in J)$. The relations (10) can be satisfied if either

$$(11) \quad (i) \quad F = 0;$$

$$(11^a) \quad (ii) \quad r_{ji} \in \text{Ann}_R(F) \text{ (for all } i \in I, j \in J).$$

The conditions (11), resp. (11^a) guarantee also that M is the cartesian product.

2. If M and M' are two subproducts with the same $\{M_i, \alpha_i, F\}_{i \in I}$, can they describe the same subproduct $M=M'$? The corresponding systems of equations are

$$(i) \quad \sum_{i \in I} r_{ji} x_i = 0 \quad (j \in J), \quad \text{resp.} \quad \sum_{i \in I} r'_{j'i} x_i = 0 \quad (j' \in J') \quad (ii).$$

If $m=(m_i)_{i \in I} \in M$, then the $\{\alpha_i(m_i)\}_{i \in I}$ have to satisfy (i) and (ii) and conversely. That means that the systems (i) and (ii) must have the same solutions $x=(x_i)_{i \in I}$. Denoting by $Y=\langle \dots, g_j, \dots \rangle$, $g_j=\sum_{i \in I} r_{ji} x_i$, $Y'=\langle \dots, g'_{j'}, \dots \rangle$, $g'_{j'}=\sum_{i \in I} r'_{j'i} x_i$, we have: if $Y=Y'$, then the solutions of the corresponding systems $\sum_i r_{ji} x_i=0(j \in J)$, resp. $\sum_i r'_{j'i} x_i=0(j' \in J')$ coincide and that implies $M=M'$. Conversely: if M and M' coincide, then the corresponding equations of any of the two systems are consequences of the equations of the other, i.e. $Y=Y'$. This means: $Y=Y'$ is the necessary and sufficient condition therefore that $M=M'$.

§ 4. The case of the special subproduct

Suppose that $M=\{M_i; \alpha_i; F; \sum_{i \in I} r_{ji} \alpha_i(m_i)=0\}_{j \in J}$ is a given subproduct; we define

$$C_i = M \cap M_i = \{m \in M | \pi_k m = 0, k \neq i\};$$

C_i is the submodule of M with elements $m=(\dots, 0, m_i, 0, \dots)$. Using the given relations, we find for the components m_i of $m \in C_i$:

$$(12) \quad r_{ji} \alpha_i(m_i) = 0 (\forall j \in J).$$

These relations can be satisfied in the following way:

(i) $F=0$; in that case $M=\prod_{i \in I} M_i$, $C_i=M_i(\forall i \in I)$;

(ii) we denote — for every fixed $i \in I$ — by S_i the subset of R defined by $S_i=\{\dots, r_{ji}, \dots\}_{j \in J}$; then there is a submodule $V_i \subset F$, defined by

$$(13) \quad V_i = (0 :_F S_i) = \{f \in F | S_i f = 0\}.$$

V_i consists of the elements $v=\alpha_i(m_i)$, satisfying the condition (12), and that means that there is a submodule $C_i=M \cap M_i$, with $\alpha_i C_i=V_i(i \in I)$; we have

$$K_i = \text{Ker } \alpha_i \subset C_i \subset M_i.$$

Let C be the submodule of M , defined by

$$(14) \quad C = \prod_{i \in I} C_i;$$

we remind that each equation $\sum_{i \in I} r_{ji} \alpha_i(m_i) = 0$ is a finite sum of expressions $r_{ji} \alpha_i(m_i)$ and for every $m_i \in C_i$, we have $r_{ji} \alpha_i(m_i) = 0$. Furthermore we define the submodules

$$(15) \quad D_i \subset M \quad (i \in I),$$

consisting of the elements $d = (d_i)_{i \in I} \in M$ with $d_i = 0$. Let $N_i = \pi_i M \subset M_i$, then

$$M/D_i \cong N_i.$$

We now suppose that M satisfies the relations

$$(16) \quad D_i \subset C \quad (\forall i \in I).$$

This implies

$$M/C \cong M/D_i/C/D_i \cong N_i/C_i;$$

using the property that $F_i = \alpha_i(N_i)$, we have

$$(17) \quad N_i/C_i \cong N_i/K_i/C_i/K_i \cong F_i/V_i \quad (i \in I).$$

This means that every M_i contains a submodule N_i such that the quotients N_i/C_i are „invariant”; i.e. if we denote M/C by $\tilde{F}$, then we have, for all $i \in I$,

$$N_i/C_i \cong \tilde{F}.$$

If $m = (\dots, m_i, \dots) \in M$, $\pi_i(m) = m_i \in N_i$, then we define $\beta_i: N_i \rightarrow \tilde{F}$ by

$$(18) \quad \beta_i(m_i) = m + C, \quad \text{if } \pi_i(m) = m_i.$$

Then β does not depend upon the choice of $m \in M$; for if $\pi_i(m) = \pi_i(m')$, then $m - m' \in D_i \subset C$, hence $m + C = m' + C$. If therefore $m = (m_i)_{i \in I} \in M$, it follows from (18), that

$$(19) \quad \dots = \beta_i(m_i) = \dots = \beta_j(m_j) = \dots = m + C;$$

this means that there is an R -module $\tilde{F}$ and epimorphisms $\beta_i: N_i \rightarrow \tilde{F} \quad (i \in I)$, such that all elements $m \in M$ satisfy the property of a special subdirect product. If, conversely, $n = (n_i)_{i \in I} \in \prod_{i \in I} N_i$ has the property, that $\dots = \beta_i n_i = \dots = \beta_j n_j = \dots$, then $n \in M$. Indeed, $\beta_i n_i = n' + C$ for some $n' \in M$, only if $\pi_i n' = n_i$; but then $n = n' \in M$. Using the notations we have defined here we have proved the

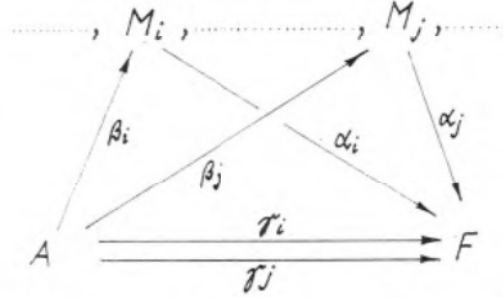
Theorem 4.1. Under the conditions $D_i \subset C \quad (i \in I)$, the subproduct $M = \{M_i; \alpha_i; F; \sum_{i \in I} r_{ji} \alpha_i(m_i) = 0\}$ is a special subdirect product $M = \times_{i \in I} N_i(\beta_i; \tilde{F})$.

If for at least one $i \in I$ we have $C_i = M \cap M_i = M_i$, then $N_i = M_i$ for all $i \in I$, and $F = M/C = 0$; in that case $M = \prod_{i \in I} M_i$. If however for at least one $i \in I$ we have $C_i \subsetneq N_i$, then this relation holds for all $i \in I$.

Conversely, suppose that the subproduct M (see (2)) has the property that M is a special subdirect product of the submodules $N_i \subset M_i$ (where $N_i = \pi_i M$) with respect to epimorphisms $\beta_i: N_i \rightarrow \tilde{F}$, then $M = \{m = (m_i)_{i \in I}, m_i \in N_i \mid \beta_i m_i = \beta_k m_k \quad (i, k \in I)\}$. For the elements $d \in D_i$, $d = (d_i)_{i \in I} \in M$, we have then $\beta_i n_i = \beta_k n_k = 0$ for all $k \in I$. Since $C = \{c = (c_i)_{i \in I} \mid \beta_i c_i = 0, \forall i \in I\}$, we have $D_i \subset C \quad (\forall i \in I)$.

§ 5. The structure of $\text{Hom}_R(A, M)$, if M is a subproduct

Let $M = \{M_i; \alpha_i; F; \sum_{i \in I} r_{ji} \alpha_i(m_i) = 0; j \in J\}$ be any subproduct, then we want to study the structure of $\text{Hom}_R(A, M)$ for any R -module A .



If $\beta \in \text{Hom}_R(A, M)$, $a \in A$, then $\beta(a) = (\dots, m_i, \dots, m_j, \dots) \in M$, and $\sum_{i \in I} r_{ji} \alpha_i(m_i) = 0$; β induces a system of R -homomorphisms

$$\beta_i: A \rightarrow M_i, \quad \beta_i = \pi_i \beta \quad (\forall i \in I),$$

where π_i is the canonical projection $\pi_i: M \rightarrow M_i (i \in I)$. Hence $\beta_i(a) = m_i (i \in I)$. Substitution in $\sum_{i \in I} r_{ji} \alpha_i(m_i) = 0$ gives

$$(2) \quad \sum_{i \in I} r_{ji} \alpha_i \beta_i(a) = 0 \quad (j \in J, \forall a \in A).$$

Defining $\alpha_i \beta_i = \gamma_i$, we see that

$$(3) \quad \sum_{i \in I} r_{ji} \gamma_i(a) = 0 \quad (j \in J, \forall a \in A).$$

This implies that if we have a subproduct M and $\beta \in \text{Hom}_R(A, M)$, then β induces a set of R -homomorphisms $\beta_i: A \rightarrow M_i$, such that (2) holds. Moreover β induces a system $(\dots, \gamma_i, \dots)$, $\gamma_i \in \text{Hom}_R(A, F)$, $\gamma_i = \alpha_i \beta_i$, $i \in I$, such that the relations

$$(4) \quad \sum_{i \in I} r_{ji} \gamma_i(a) = 0 \quad (j \in J)$$

hold for all $a \in A$. Conversely: suppose we have a set of R -homomorphisms $\beta_i \in \text{Hom}_R(A, M_i)$, $i \in I$, such that for all $a \in A$ we have the identities $\sum_{i \in I} r_{ji} \alpha_i \beta_i(a) = 0$; then the system $(\dots, \beta_i, \dots)$ determines in a unique way a homomorphism $\beta: A \rightarrow M$ by $\beta(a) = (\dots, \beta_i(a), \dots)$ and this element belongs to M , since (2) is satisfied. Moreover we learn from (4), that — for a certain $j \in J$ — we have

$$r_{j1} \gamma_1(a) + r_{j2} \gamma_2(a) + \dots + r_{jk_j} \gamma_{k_j}(a) = 0 \quad (\forall a \in A),$$

i.e.

$$r_{j1} \gamma_1 + r_{j2} \gamma_2 + \dots + r_{jk_j} \gamma_{k_j} = 0,$$

(using the fact that R is commutative!); that means — for all $j \in J$ — we have the identities

$$(5) \quad \sum_{i \in I} r_{ji} \gamma_i = 0 \quad (j \in J).$$

The result can be expressed as follows:

Theorem 5.1. *If M is a subproduct, defined by (1), and A is any R -module, then $\text{Hom}_R(A, M)$ is a subproduct of the $\text{Hom}_R(A, M_i)$, determined by*

$$\text{Hom}_R(A, M) \cong \{ \text{Hom}_R(A, M_i); (\alpha_i)_*; \text{Hom}_R(A, F); \sum_{i \in I} r_{ji} (\alpha_i)_* = 0; j \in J \},$$

where $(\alpha_i)_* = \alpha_i \beta_i$.

The result is an exchange of the operation Hom with the operation of taking subproducts; in fact, it means that any $\beta \in \text{Hom}_R(A, M)$ is determined by a set $(\beta_i)_{i \in I}$, $\beta_i \in \text{Hom}_R(A, M_i)$, such that the relations $\sum_{i \in I} r_{ji} \alpha_i \beta_i = 0$ ($j \in J$) are satisfied.

Remark. We know that the structure of a subdirect product of n modules M_i ($n > 2$) is rather complicated. Prof. L. FUCHS sent me the following remark, proving that a subdirect product of finitely many modules can be described by a subproduct.

Let M be a subdirect product $M = \bigtimes_{i=1}^n M_i$ of the finitely many modules $M_1, \dots, M_n$. Then define $F = \bigoplus_{i=1}^n M_i / M$ and let $\alpha_i: M_i \rightarrow F$ be the maps defined by $\alpha_i: m_i \mapsto m_i + M$. Then M can be recaptured by using the single equation $\sum_{i=1}^n \alpha_i m_i = 0$.

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