

Multiplicative congruences on matrixsemigroups

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Let R be a ring with unity and $M_n(R)$, $n > 1$, the matrixring with unity E over the ring R . The group of units of R and $M_n(R)$ will be denoted by $G(R)$ and $GL_n(R)$ and the elementary matrices of $M_n(R)$ by $D_q(a)$, $T_{pq}(b)$ and P_{pq} ; $p, q \in \{1, 2, \dots, n\}$. The matrix E_{pq} represents a matrix with 1 as (p, q) -component and every other component zero.

Let $\mathcal{C}$ be the set of all multiplicative semigroup congruences δ on $M_n(R)$ such that the factorsemigroup $M_n(R)/\delta$ is commutative and $\mathcal{D}$ the set of all multiplicative semigroup congruences σ on $M_n(R)$ such that $T_{pq}(b) \sim E(\sigma)$ for all $p, q \in \{1, 2, \dots, n\}$ and for all b of R .

Lemma 1. $T_{pq}(b)$ is a groupcommutator of $GL_n(R)$ for all $b \in R$ and for $p \notin \{q-1, q+1\}$.

PROOF. It can be checked that for $p \in \{q+2, q+3, \dots\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

in which

$$X = T_{p-1,q}(-b) + E_{p,p-1}$$

$$Y = T_{p-1,q}(b)$$

and for $p \in \{\dots, q-3, q-2\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

where

$$X = T_{p+1,q}(-b) + E_{p,p+1}$$

$$Y = T_{p+1,q}(b)$$

Lemma 2. Let R be a ring with an element a such that $\{a, (1-a)\} \subseteq G(R)$ then $T_{q-1,q}$ and $T_{q+1,q}$ are groupcommutators in $GL_n(R)$

PROOF. It can be verified that for $p \in \{q-1, q+1\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

where

$$X = T_{pq}((1-a)^{-1} \cdot b)$$

$$Y = D_p(a)$$

From lemma 1 and lemma 2 one deduce immediately the following

Theorem 1. *If R is a ring containing an element a such that $\{a, (1-a)\} \subseteq G(R)$, then $\mathcal{C} \subseteq \mathcal{D}$.*

In the following we consider only rings R such that the multiplicative semigroup R and the matrixsemigroup $M_n^\circ(R)$ are semigroups with separating group part [1], p. 256. We denote by

- $D_q(R)$, D , P and $TL_n(R)$ respectively the sets $\{D_q(r) \mid r \in R\}$, $\bigcup D_q(R)$, $\{P_{pq} \mid p, q \in \{1, 2, \dots, n\}\}$ and $\{T_{pq}(r) \mid r \in R \text{ and } p, q \in \{1, 2, \dots, n\}\}$
- $M_n^\circ(R)$ the multiplicative subsemigroup of $M_n(R)$ generated by the set $D \cup P \cup TL_n(R)$
- $GL_n^\circ(R)$ the group of units of $M_n^\circ(R)$ with commutatorgroup $CL_n^\circ(R)$.
- Φ the empty set.

Lemma 3. *$TL_n(R)$ is an admissible set, [2] p 178] in $M_n^\circ(R)$ and an invariant subgroup of $GL_n^\circ(R)$ containing the commutatorgroup $CL_n^\circ(R)$ of $GL_n^\circ(R)$.*

PROOF. We have to prove that for all matrices A of $GL_n^\circ(R)$

$$(1) \quad ATL_n(R) = TL_n(R) A$$

Since

$$D_q(r) = P_{1q} \cdot D_1(r) \cdot P_{1q}$$

and

$$P_{pq} = T_{pq}(1) \cdot T_{qp}(-1) \cdot T_{pq}(1) \cdot D_p(-1)$$

(1) reduces to

(2) for all r belonging to $G(R): D_1(r) TL_n(R) = TL_n(R) D_1(r)$ which follows from the identities

$$(3) \quad D_p(a) \cdot T_{rs}(b) = T_{rs}(b) \cdot D_p(a) \quad \text{if } p \neq r \text{ and } p \neq s$$

$$D_p(a) \cdot T_{ps}(b) = T_{ps}(a \cdot b) \cdot D_p(a) \quad \text{if } p \neq s$$

$$D_p(a) \cdot T_{rp}(b) = T_{rp}(b \cdot a^{-1}) \cdot D_p(a) \quad \text{if } r \neq p$$

For all $p, q \in \{1, 2, \dots, n\}$ and for all $a, b \in G(R)$ we have

$$P_{pq} TL_n(R) = D_p(-1) TL_n(R)$$

$$D_p(a) TL_n(R) = D_q(a) TL_n(R)$$

$$D_p(a \cdot b) TL_n(R) = D_p(b \cdot a) TL_n(R)$$

and hence, for all $A, B \in GL_n^\circ(R)$:

$$ABTL_n(R) = BATL_n(R)$$

Suppose that

$$ATL_n(R) B \cap TL_n(R) \neq \emptyset \quad \text{with } A, B \in M_n^\circ(R)$$

or, equivalently

$$ATL_n(R) \cap TL_n(R) \neq \emptyset \quad \text{with} \quad A, B \in GL_n^\circ(R)$$

or,

$$ABTL_n(R) \cap TL_n(R) \neq \emptyset \quad \text{with} \quad A, B \in GL_n^\circ(R)$$

Then

$$ABTL_n(R) \subseteq TL_n(R)$$

or

$$ATL_n(R) \subseteq TL_n(R),$$

from which we may conclude that $TL_n(R)$ is an admissible set in the semigroup $M_n^\circ(R)$.

It is well known, see [2], p. 179, that every semigroupcongruence δ on the multiplicative semigroup $M_n^\circ(R)$ which admits $TL_n(R)$ satisfies the relation

$$(4) \quad \alpha C^* T \subseteq \delta \subseteq \beta C$$

in which

$$\alpha = (TL_n(R) \times TL_n(R)) \cup \{(A, A) \mid A \in M_n^\circ(R) \setminus TL_n(R)\}$$

$$\beta = (TL_n(R) \times TL_n(R)) \cup (M_n^\circ(R) \setminus TL_n(R) \times M_n^\circ(R) \setminus TL_n(R))$$

and

$$\alpha C^* T = \text{the unique minimal congruence containing } \alpha$$

$$\beta C = \text{the unique maximal congruence contained in } \beta.$$

Definition. A semigroupconvergence δ satisfying the relation (4) will be called a *T-congruence* on the multiplicative semigroup $M_n^\circ(R)$.

Theorem 2. If δ is a *T-congruence* on the multiplicative semigroup $M_n^\circ(R)$ then the factorsemigroup $M_n^\circ(R)/\delta$ is commutative and there exists a semigroupcongruence ϱ on the multiplicative semigroup R such that $M_n^\circ(R)/\delta$ is isomorphic with R/ϱ .

PROOF. If $\bar{A}$ and $\bar{B}$ represent any two elements of $M_n^\circ(R)/\delta$ then $\bar{A}$ contains a matrix of the form $D_1(a)$ and $\bar{B}$ contains a matrix of the form $D_n(b)$ for which

$$\overline{D_1(a)} \cdot \overline{D_n(b)} = \overline{D_n(b)} \cdot \overline{D_1(a)}$$

and hence

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}.$$

The restriction of the semigroupcongruence δ to the subsemigroup $D_1(R)$ of $M_n^\circ(R)$ induces a congruence ϱ on the multiplicative semigroup R of the ring R since the multiplicative semigroup R is isomorphic with $D_1(R)$. The stated semigroupisomorphism follows from the fact that each class of $M_n^\circ(R)/\delta$ contains elements of $D_1(R)$.

If $M_n^\circ(R) \neq M_n(R)$, then one can introduce the following equivalence relation σ on $M_n(R)$:

$$\begin{aligned} X \sim Y(\sigma) &\Leftrightarrow \sum_{\pi} \operatorname{sgn} \pi x_{1, \pi(1)}, x_{2, \pi(2)}, \dots, x_{n, \pi(n)} \sim \\ &\sim \sum_{\pi} \operatorname{sgn} \pi' y_{1, \pi'(1)}, y_{2, \pi'(2)}, \dots, y_{n, \pi'(n)}(\varrho) \dots \end{aligned}$$

in which π and π' are permutations on the set $\{1, 2, \dots, n\}$ and ϱ the semigroup-congruence on R of theorem 2.

Theorem 3. *If the associated semigroupcongruence ϱ on R of a T -congruence δ on $M_n^\circ(R)$ is also a ringcongruence on R , then the relation σ on $M_n(R)$ is a semigroupcongruence.*

PROOF. The result follows from the fact that the factorring R/ϱ is commutative.

Examples

1. Matrixrings over division rings.

The multiplicative semigroup $M_n^\circ(L)$ over a division ring L is a semigroup with separating group part. Moreover $M_n(L) = M_n^\circ(L)$ and by theorem 1, the set $TL_n(L)$ is an invariant subgroup of the group $GL_n(L)$ containing the commutator-group $CL_n(L)$ of $GL_n(L)$ which is a well-known result since for $n \neq 2$ and for $n=2$ with $L \neq \{0, 1\}$ one has $TL_n(L) = CL_n(L)$. For $n=2$ together with $L = \{0, 1\}$ one has $CL_2(\{0, 1\}) \subseteq TL_2(\{0, 1\})$. Moreover, there exists a semigroupcongruence δ on $M_n^\circ(L)$ which admits $TL_n(L)$. By theorem 2, there exists a nontrivial semigroupcongruence ϱ on the division ring L such that $M_n(L)/\delta$ is semigroupisomorphic with L/ϱ . Since L/ϱ is a group with zero, the factorsemigroup $M_n(L)/\delta$ will be also a group with zero and with unity $TL_n(L)$. By the results of P. Dubreil, [2], p. 182, δ must be unique and be equal to

$$R_{TL_n(L)} = \{(A, B) \in \text{cart}^2 M_n(L) \mid AX \in TL_n(L) \Leftrightarrow BX \in TL_n(L), \quad \forall X \in M_n(L)\}$$

It is clear that, unless $n=2$ together with $L = \{0, 1\}$, δ will be the semigroupcongruence determined by the determinantmorphism defined by J. Dieudonné [3], for which $GL_n(L)/CL_n(L)$ is groupisomorphic with $G(L)/C$, in which $CL_n(L)$ represents the commutatorgroup of the group $GL_n(L)$ and C the commutator-group of the group $G(L)$. In the case $n=2$ together with $L = \{0, 1\}$ it can be proved that $M_2(\{0, 1\})/\delta$ is semigroupisomorphic with the group with zero $\{0, 1, x \mid x^2 = 1\}$.

2. Matrixrings over commutative rings.

Let R_c be any commutative ring. The multiplicative semigroups R_c and $M_n^\circ(R_c)$ are semigroups with separating group part. By theorem 1, the subset $TL_n(R_c)$ of the matrixring $M_n(R_c)$ over a commutative ring R_c is an invariant subgroup of the group $GL_n^\circ(R_c)$ containing the commutatorgroup $CL_n^\circ(R_c)$ of $GL_n^\circ(R_c)$ and $TL_n(R_c)$ is an admissible set in $M_n^\circ(R_c)$. Every T -convergence δ admitting $TL_n(R_c)$ satisfies the relation $\alpha C^* T \subseteq \delta \subseteq \beta C$. The T -congruences $\alpha C^* T$ and βC on $M_n^\circ(R_c)$ can be characterised as follows.

Let $\delta_{\det_r}$ denote the semigroupcongruence associated with the restriction $\det_r$ of the determinant on $M_n^\circ(R_c)$.

Proposition 1: $\alpha C^* T = \delta_{\det_r}$

PROOF. Clearly $TL_n(R_c) \subseteq \det_r^* \{1\} = \{M \in M_n^\circ(R_c) \mid \det M = 1\}$. Conversely, let A be a matrix of $M_n^\circ(R_c)$ such that $\det_r A = 1$. From (3) it follows that A can be written as a product of $T_{pq}(b)$ -matrices and a diagonalmatrix $\text{diag}(a_{11}, a_{22}, \dots, a_{nn})$

with $\prod_{i=1}^n a_{ii} = 1$. This diagonal matrix will be called an associated diagonal matrix D_A of the matrix A of $M_n^\circ(R_c)$. This matrix D_A also belongs to $TL_n(R_c)$ since D_A can be written as a product of matrices of the form $D_i(r) \cdot D_j(r^{-1})$ for which $D_i(r) \cdot D_j(r^{-1}) = T_{ij}(r) \cdot T_{ji}(1-r^{-1}) \cdot T_{ij}(-1) \cdot T_{ji}(1-r)$. Therefore A belongs to $TL_n(R_c)$. This means that $\alpha C^* T \subseteq \delta \det_r \subseteq \beta C^* T$ and it remains to prove that $\delta \det_r \subseteq \alpha C^* T$. Let $(A, B) \in \text{cart}^2 M_n^\circ(R_c)$, then from $A \sim B(\delta \det_r)$ it follows that $\det_r A = \det_r B$ and $D_1(\det_r A) = D_1(\det_r B)$. If D_A and D_B represent respectively associated diagonal matrices of the matrices A and B then clearly $D_A \sim A(\alpha C^* T)$ and $D_B \sim B(\alpha C^* T)$. Since also $D_A \sim D_1(\det_r A)(\alpha C^* T)$ and $D_B \sim D_1(\det_r B)(\alpha C^* T)$ we obtain $D_A \sim D_B(\alpha C^* T)$ and finally $A \sim B(\alpha C^* T)$.

Proposition 2. If R_c is a commutative ring then the unique maximal congruence admitting $TL_n(R_c)$ of $M_n(R_c)$ equals the Dubreil congruence associated with the set $TL_n(R_c)$.

PROOF. Since $TL_n(R_c)$ is a normal divisor of $GL_n^\circ(R_c)$ and $\text{co}(GL_n^\circ(R_c))$ a primesemigroupideal of $M_n^\circ(R_c)$, the partition determined by the cosets of $TL_n(R_c)$ in $GL_n^\circ(R_c)$ and the set $\text{co}(GL_n^\circ(R_c))$ determines a congruence δ on $M_n^\circ(R_c)$ such that $M_n^\circ(R_c)/\delta$ will be a group with zero. Therefore δ must be the Dubreil congruence associated with the set $TL_n(R_c)$. It is the unique maximal congruence admitting $TL_n(R_c)$ since any congruence which is greater than δ will be the trivial congruence on $M_n^\circ(R_c)$.

Corollary 1. Any T -congruence on $M_n^\circ(R_c)$ is greater than the determinant congruence $\delta \det_r$ and smaller than the Dubreil congruence associated to the set $TL_n(R_c)$ in $M_n^\circ(R_c)$.

Corollary 2. If R_E denotes an Euclidean ring then $M_n^\circ(R_E) = M_n(R_E)$ and therefore one has : any T -convergence on $M_n(R_E)$ is greater than the determinant congruence $\delta \det$ and smaller than the Dubreil congruence associated to the set $TL_n(R_c)$ in $M_n(R_E)$.

The multiplicative congruence ϱ on R corresponding with the multiplicative congruence $\delta \det_r$ on $M_n^\circ(R_c)$ consist of the diagonal Δ of $\text{cart}^2 R$ and therefore is a ring congruence on R_c . By theorem 3 the relation

$$\begin{aligned} X \sim Y(\sigma) &\Leftrightarrow \sum_{\pi} \text{sgn } \pi x_{1, \pi(1)}, x_{2, \pi(2)}, \dots, x_{n, \pi(n)} = \\ &= \sum_{\pi'} \text{sgn } \pi' x_{1, \pi'(1)} \cdot x_{2, \pi'(2)}, \dots, x_{n, \pi'(n)} \end{aligned}$$

must be a multiplicative congruence on $M_n(R_c)$ which is indeed the case since it is the multiplicative congruence associated with the determinant on $M_n(R_c)$.

References

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