

On generalized absolute Cesaro summability factors

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Let Σa_n be a given series with partial sums s_n . By $\sigma_n^{(\alpha)}$ we denote the n -th Cesaro means of order α of the sequences $\{s_n\}$. If the series $\Sigma n^{k-1}|\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^k$ ($k \geq 1; \alpha > -1$) is convergent, we say that the series Σa_n is absolute summable (C, α) with index k or simply summable $|C, \alpha|_k$.

A sequence $\{\lambda_n\}$ is said to be a $|C, \alpha|_k$ summability factor of the series Σa_n if multiplying it term by term with λ_n , the factored series is $|C, \alpha|_k$ summable.

We have the following theorems.

Theorem 1. Let $k \geq 1, 0 \leq \alpha \leq 1$ and let $\{\lambda_n\}$ be a positive non-increasing and convex sequence, $\{\gamma_n\}$ be a positive non-decreasing sequence such that $\Delta \gamma_n = O(\gamma_n/n)$ ($n = 1, 2, \dots; \mu = n, n+1, \dots, 2n$). If

$$(1) \quad \sum_{n=1}^{\infty} \lambda_n^k |\Delta \gamma_n| < \infty$$

and

$$(2) \quad \sum_{v=1}^n v^{k-1} |\sigma_v^{(\alpha)} - \sigma_{v-1}^{(\alpha)}|^k = O(\gamma_n)$$

then the sequence $\{\lambda_n\}$ is a $|C, \alpha|_k$ summability factor of the series Σa_n .

The special case $k=1, \gamma_n = \log n$ and $\lambda_n = (\log n)^{-1-\varepsilon}$ ($\varepsilon > 0$) was proved by SUNOUCHI [7]. Using a result of NÉMETH (see Lemma 2) we can prove that

$$\sum_{v=1}^n v^{k-1} |\sigma_v^{(\alpha)} - \sigma_{v-1}^{(\alpha)}|^k = O\left(\sum_{v=1}^n \frac{|s_v|^k}{v^{k(\alpha-1)+1}}\right),$$

so we have the following.

Corollary. If the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ are satisfying the conditions of Theorem 1 and

$$\sum_{v=1}^N \frac{|s_v|^k}{v^{k(\alpha-1)+1}} = O(\gamma_n)$$

then the series $\Sigma \lambda_n a_n$ is $|C, \alpha|_k$ summable.

The special case $\alpha=1$ and $\gamma_n = \log n$ was proved by MAZHAR [3] and MISHRA [4].

The next theorem gives a result concerning the $|C, \alpha|_k$ summability of negative order α , too.

Theorem 2. Let $k \geq 1$; $-\frac{1}{k} < \alpha \leq 1$ and let $\{\lambda_n\}$ be a positive convex sequence such that

$$(3) \quad \sum_{n=1}^{\infty} \frac{\lambda_n^k}{n^{k(\alpha-1)+1}} < \infty.$$

If

$$(4) \quad \sum_{v=1}^n \left| \frac{1}{v} \sum_{\mu=1}^v \mu a_{\mu} \right|^k = O(n)$$

then the sequence $\{\lambda_n\}$ is a $|C, \alpha|_k$ summability factor of the series $\sum a_n$.

Finally we have two results for Fourier series.

Theorem 3. If $\{\lambda_n\}$ satisfies the conditions of Theorem 2 then $\{\lambda_n\}$ is a $|C, \alpha|_k$ summability factor of Fourier series of any integrable function almost everywhere.

The special case $k=1$ and $\alpha=1$ was proved by CHOW [1] and SUNOCHI [6]. For $f(x) \in L^p$ we define

$$\omega_p(t, f) = \sup_{0 < h \leq t} \left\{ \int_{-\pi}^{\pi} |f(x+h) + f(x-h) - 2f(x)|^p dx \right\}^{1/p}.$$

Theorem 4. Let $f(x) \in L^p$ ($1 < p \leq 2$), $1 \leq k < 2$ and let $\{\mu_n\}$ be a non-increasing sequence tending to zero and satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{n \left(\sum_{v=1}^n \mu_v \right)^{\frac{2k}{2-k}}} < \infty.$$

If

$$\sum_{n=1}^{\infty} \mu_n \omega_p \left(\frac{1}{n}, f \right) < \infty$$

then the Fourier series of $f(x)$ is summable $|C, \alpha|_k$ almost everywhere for $\alpha > \frac{1}{p}$.

The special case $k=1$ was proved in [8]. The proof of Theorem 4 runs similarly to that of Theorem I of [8], we only have to use Lemma 1 instead of Lemma 4 of [8], so we omit the proof of Theorem 4.

We need the following lemmas. Generalizing and a little bit modifying a Chow's lemma ([2], Lemma 4) we obtain

Lemma 1. Let $k \geq 1$, $0 \leq \alpha \leq 1$ and let $\{\lambda_n\}$ be a positive sequence such that the sequence $\{|\Delta \lambda_n|\}$ is¹⁾ non-increasing. If

$$(5) \quad \sum_{n=1}^{\infty} \frac{\lambda_n^k |t_n^{(\alpha)}|^k}{n} < \infty \quad (t_n^{(\alpha)} = n(\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}))$$

¹⁾ $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$

and

$$(6) \quad \sum_{n=1}^{\infty} n^{k-1} |\Delta \lambda_n|^k |l_n^{(\alpha)}|^k < \infty$$

then²⁾ the series $\sum \lambda_n a_n$ is $|C, \alpha|_k$ summable.

We require the following inequality of NÉMETH ([5]) as

Lemma 2. Let $d_n \geq 0$ and $l_n > 0$ ($n=1, 2, \dots$) be given sequences. If the triangular matrix $M=(C_{n,v})$ satisfies the following conditions $C_{n,v} \geq 0$ ($v < n$), $C_{n,v} = 0$ ($v \geq n$) and $0 \leq C_{m,v} \leq C \cdot C_{n,v}$ ($0 < v \leq n \leq m$) C denotes a positive absolute constant, then for any $k \geq 1$

$$\sum_{n=1}^{\infty} l_n \left(\sum_{v=1}^n C_{n,v} d_v \right)^k \leq C^{k(k-1)} k^k \sum_{n=1}^{\infty} l_n^{1-k} \left(\sum_{v=n}^{\infty} l_v C_{v,n} \right)^k d_n^k.$$

Using the Abel transformation it is easy to prove the following lemmas.

Lemma 3. If $k \geq 1$ and $\{b_n\}$ is a non-increasing sequence such that $\sum_{n=1}^{\infty} b_n < \infty$, then the series $\sum_{n=1}^{\infty} n^{k-1} b_n^k$ is convergent and for every $v=1, 2, \dots$

$$(7) \quad \sum_{n=4v+1}^{\infty} n^{k-1} b_n^k = O\left(\left\{\sum_{n=v}^{\infty} b_n\right\}^k\right)$$

hold.

Lemma 4. Let $\{b_n\}$ be a non-increasing sequence such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\{C_n\}$ be a sequence of positive numbers such that $nC_{\mu} = O\left(\sum_{i=1}^n c_i\right)$, where $n=1, 2, \dots$; $\mu=n, n+1, \dots, 2n$. If $k \geq 1$ and

$$(8) \quad \sum_{n=1}^{\infty} \left(\sum_{m=n}^{\infty} b_m \right)^k C_n < \infty$$

then the series $\sum_{n=1}^{\infty} n^k b_n^k C_n$ is convergent.

Finally we shall use the following obvious facts as lemmas.

Lemma 5. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\varrho_n\}$ be given positive sequences such that $\{\alpha_n\}$ non-increasing, $\{\varrho_n\}$ non-decreasing. If $\sum_{v=1}^n \alpha_{v+1} |\Delta \varrho v| = O(1)$ and $\sum_{v=1}^n \beta_v = O(\varrho_n)$ then $\sum_{v=1}^n \alpha_v \beta_v = O(1)$.

²⁾ If $\alpha=0$ the condition (6) is superfluous.

Lemma 6. Let $\alpha_n \geq 0$ and $\beta_n \geq 0$ be given sequences. If $\sum_{n=1}^{\infty} \alpha_n < \infty$, $C_1 \alpha_{2^{n+1}} \leq \alpha_{2^n + v} \leq C_2 \alpha_{2^n}$ and $\sum_{\mu=1}^n \beta_{\mu} = O(n)$ ($n=1, 2, \dots$; $v=1, 2, \dots, 2^n$) then $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$.

PROOF of Theorem 1.

First using Lemma 5 with $\alpha_n = \lambda_n^k$, $\beta_n = n^{k-1} |\sigma_n^{(x)} - \sigma_{n-1}^{(x)}|^k$ and $\varrho_n = \gamma_n$ by (1) and (2) we obtain (5).

Secondly we use Lemma 5 $\alpha_n = (\Delta \lambda_n)^k$, $\beta_n = n^{2k-1} |\sigma_n^{(x)} - \sigma_{n-1}^{(x)}|^k$ and $\varrho_n = n^k \gamma_n$. By (2) we have that $\sum_{v=1}^n \beta_v = O(\varrho_n)$. If we can show that

$$(9) \quad \sum_{n=1}^{\infty} (\Delta \lambda_{n+1})^k |\Delta(n^k \gamma_n)| < \infty$$

then, by Lemma 5, we have (6). Finally we apply Lemma 1 and have Theorem 1. Now we prove (9)

$$\begin{aligned} \sum_{n=1}^{\infty} (\Delta \lambda_{n+1})^k |\Delta(n^k \gamma_n)| &\leq K(k) \sum_{n=1}^{\infty} n^k (\Delta \lambda_n)^k |\Delta \gamma_n| + \\ &+ K(k) \sum_{n=1}^{\infty} n^{k-1} (\Delta \lambda_n)^k \gamma_n \equiv S_1 + S_2. \end{aligned}$$

The constant K is depending on its argument only.

Using Lemma 4 with $b_n = \Delta \lambda_n$ and $C_n = |\Delta \gamma_n|$, by (1) and (8) we have that S_1 is finite.

Applying Lemma 3 with $b_n = \Delta \lambda_n$ and $C_n = |\Delta \gamma_n|$ we get

$$\begin{aligned} S_2 &\leq K(k) \left(\sum_{n=2}^{\infty} n^{k-1} (\Delta \lambda_n)^k \sum_{v=1}^{n-1} |\Delta \gamma_v| + \gamma_1 \sum_{n=2}^{\infty} n^{k-1} (\Delta \lambda_n)^k \right) = \\ &= K(k) \left(\sum_{v=1}^{\infty} |\Delta \gamma_v| \sum_{n=v+1}^{\infty} n^{k-1} (\Delta \lambda_n)^k + O(1) \right) = \\ &= K(k) \left(\sum_{v=1}^{\infty} |\Delta \gamma_v| \sum_{n=v+1}^{4v} n^{k-1} (\Delta \lambda_n)^k + \sum_{v=1}^{\infty} |\Delta \gamma_v| \sum_{n=4v+1}^{\infty} n^{k-1} (\Delta \lambda_n)^k \right) \equiv \\ &\equiv S_{21} + S_{22}. \end{aligned}$$

Since $\{\Delta \lambda_n\}$ is non-increasing we have $S_{21} = O(S_1)$.

Applying Lemma 3 again, by (7) and (1), we can see that S_{22} is also finite so, our proof is complete.

PROOF of Theorem 2. Let us denote by $\sigma_n^{(x)}(\lambda)$ the n -th (C, α) mean of the series $\sum \lambda_n a_n$. Using the symbols

$$A_n^{(x)} = \binom{n+\alpha}{n} \quad \text{and} \quad L_{n,v}^{(x)} = \frac{A_{n-v}^{(x)}}{A_n^{(x)}} \cdot \frac{v\alpha}{(n+1-v)(n+1+\alpha)} \quad (v=1, \dots, n),$$

by Abel's transformation we have that for $n \geq 3$

$$\begin{aligned}
 (10) \quad \sigma_n^{(\alpha)}(\lambda) - \sigma_{n-1}^{(\alpha)}(\lambda) &= \sum_{v=1}^{n-1} L_{n-1,v}^{(\alpha)} \lambda_v a_v + \frac{1}{A_n^{(\alpha)}} \lambda_n a_n = \\
 &= \sum_{v=1}^{n-2} \left(\sum_{\mu=1}^v \mu a_\mu \right) \left(\frac{L_{n-1,v}^{(\alpha)}}{v} \lambda_v - \frac{L_{n-1,v+1}^{(\alpha)}}{v+1} \lambda_{v+1} \right) + \\
 &+ \left(\sum_{\mu=1}^{n-1} \mu a_\mu \right) \left(\frac{L_{n-1,n-1}^{(\alpha)}}{n-1} \lambda_{n-1} - \frac{\lambda_n}{n A_n^{(\alpha)}} \right) + \left(\sum_{\mu=1}^n \mu a_\mu \right) \frac{\lambda_n}{n A_n^{(\alpha)}}.
 \end{aligned}$$

A simple computation shows that

$$(11) \quad \frac{L_{n-1,v}^{(\alpha)}}{v} \lambda_v - \frac{L_{n-1,v+1}^{(\alpha)}}{v+1} \lambda_{v+1} = \frac{\alpha}{(n+\alpha) A_{n-1}^{(\alpha)}} \left(\frac{A_{n-v-1}^{(\alpha)}}{n-v} \Delta \lambda_v + \frac{(\alpha-1) A_{n-v-2}^{(\alpha)}}{(n-v+1)(n-v)} \lambda_{v+1} \right)$$

and

$$(12) \quad \frac{L_{n-1,n-1}^{(\alpha)}}{n-1} \lambda_{n-1} - \frac{\lambda_n}{n A_n^{(\alpha)}} = \frac{1}{(n+\alpha) A_{n-1}^{(\alpha)}} (\alpha \Delta \lambda_{n-1} + (\alpha-1) \lambda_n).$$

Looking at (10), (11) and (12) we get the estimation

$$\begin{aligned}
 (13) \quad \sum_{n=1}^{\infty} n^{k-1} |\sigma_n^{(\alpha)}(\lambda) - \sigma_{n-1}^{(\alpha)}(\lambda)|^k &\leq \\
 &\leq C(\alpha, k) \sum_{n=2}^{\infty} \frac{1}{n^{k\alpha+1}} \left(\sum_{v=1}^{n-1} (n-v)^{\alpha-1} \left| \sum_{\mu=1}^v \mu a_\mu \right| s \lambda_v \right)^k + \\
 &+ C^*(\alpha, k) \sum_{n=3}^{\infty} \frac{1}{n^{k\alpha+1}} \left(\sum_{v=1}^{n-2} (n-v-1)^{\alpha-2} \left| \sum_{\mu=1}^v \mu a_\mu \right| \lambda_{v+1} \right)^k + \\
 &+ C(\alpha, k) \sum_{n=1}^{\infty} \frac{1}{n^{k\alpha+1}} \left| \sum_{\mu=1}^n \mu a_\mu \right|^k \lambda_n^k \equiv S_3 + S_4 + S_5.
 \end{aligned}$$

First we shall consider S_5 . Since

$$S_5 = C(\alpha, k) \sum_{n=1}^{\infty} \frac{\lambda_n^k}{n^{k(\alpha-1)+1}} \left| \frac{1}{n} \sum_{\mu=1}^n \mu a_\mu \right|^k,$$

by (3), (4) and Lemma 6 we get that S_5 is finite.

We shall now estimate S_3 . We put $C_{n,v} = (n-v)^{\alpha-1}$ for $v < n$, $d_n = \left| \sum_{\mu=1}^n \mu a_\mu \right| \Delta \lambda_n$, $l_n = n^{-k\alpha-1}$ and applying Lemma 2 we have

$$S_3 \leq C(\alpha, k) \sum_{n=2}^{\infty} \left(\frac{1}{n^{k\alpha+1}} \right)^{1-k} \left(\sum_{v=n+1}^{\infty} \frac{(v-n)^{\alpha-1}}{v^{k\alpha+1}} \right)^k \left| \sum_{\mu=1}^n \mu a_\mu \right|^k (\Delta \lambda_n)^k$$

and from this a standard computation shows that

$$S_3 \equiv C(\alpha, k) \sum_{n=2}^{\infty} \frac{(\Delta \lambda_n)^k}{n^{k(\alpha-1)+1}} \left| \sum_{\mu=1}^n \mu a_{\mu} \right|^k.$$

If we can show that the series

$$\sum_{n=1}^{\infty} \frac{(\Delta \lambda_n)^k}{n^{k(\alpha-2)+1}}$$

is convergent, then by (4) and Lemma 6 we get S_3 is finite. By condition (3) we have

$$\begin{aligned} \sum_{n=5}^{\infty} \frac{(\Delta \lambda_n)^k}{n^{k(\alpha-2)+1}} &\leq \sum_{m=2}^{\infty} (2^{m+1} \Delta \lambda_{2^m})^k \sum_{n=2^{m+1}}^{2^{m+1}-1} \frac{1}{n^{k(\alpha-1)+1}} \leq \\ &\leq C(k) \sum_{m=2}^{\infty} \lambda_{2^{m-1}}^k \sum_{n=2^{m-2}+1}^{2^{m-1}-1} \frac{1}{n^{k(\alpha-1)+1}} \leq C(k) \sum_{n=1}^{\infty} \frac{\lambda_n^k}{n^{k(\alpha-1)+1}} < \infty. \end{aligned}$$

It remains to estimate S_4 . Considering (10) and (11) we can choose $C^*(1, k) = 0$ in (13), so we can assume that $-\frac{1}{k} < \alpha < 1$. Putting $C_{n,v} = (n-v-1)^{\alpha-2}$ for $v < n-1$ and zero for $v = n-1$, $d_n = \left| \sum_{\mu=1}^n \mu a_{\mu} \right| \lambda_{n+1}$, $l_n = n^{-k\alpha-1}$ and applying Lemma 2 we have

$$\begin{aligned} S_4 &\leq C(\alpha, k) \sum_{n=3}^{\infty} \left(\frac{1}{n^{k\alpha+1}} \right)^{1-k} \left(\sum_{v=n+2}^{\infty} \frac{(v-n-1)^{\alpha-2}}{v^{k\alpha+1}} \right)^k \left| \sum_{\mu=1}^n \mu a_{\mu} \right|^k \lambda_{n+1}^k \leq \\ &\leq C(\alpha, k) \sum_{n=3}^{\infty} \frac{1}{n^{k\alpha+1}} \left| \sum_{\mu=1}^n \mu a_{\mu} \right|^k \lambda_{n+1}^k \leq C(\alpha, k) S_5. \end{aligned}$$

Our proof is complete.

PROOF of Theorem 3. Let

$$(14) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)$$

and let $s_n(x)$ and $\sigma_n(x)$ denote the n -th partial sum and n -th $(C, 1)$ mean of the Fourier series (14), respectively. We have by Zygmund's theorem ([9], vol. II. p. 184.) that

$$\sum_{v=1}^n |s_v(x) - \sigma_v(x)|^k \leq C(x, k) \cdot n$$

almost everywhere. Using the identity

$$s_v(x) - \sigma_v(x) = \frac{1}{v+1} \sum_{\mu=1}^v \mu A_{\mu}(x)$$

by Theorem 2 we get Theorem 3.

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