

Non-additive ring and module theory II. $\mathcal{C}$ - Categories, $\mathcal{C}$ - Functors and $\mathcal{C}$ - Morphisms

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Consider the example $(\text{Cat}, \times, \mathbf{1})$ of a monoidal category. A monoid in this category is a (small) category $\mathcal{C}$ together with functors $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $\mathcal{I}: \mathbf{1} \rightarrow \mathcal{C}$ (where we use the notation $\mathcal{I}(e) = I$) such that the diagrams

$$(*) \quad \begin{array}{ccccc} \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{\mathcal{C} \times \otimes} & \mathcal{C} \times \mathcal{C} & \xrightarrow{\mathcal{C} \times \mathcal{I}} & \mathcal{C} \times \mathcal{C} \\ \otimes \times \mathcal{C} \downarrow & & \downarrow \otimes & \searrow \mathcal{I} \times \mathcal{C} & \downarrow \otimes \\ \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \end{array}$$

commute. This means $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for all $A, B, C \in \mathcal{C}$ and $A \otimes I = A = I \otimes A$ for all $A \in \mathcal{C}$. These identities are natural transformations in A, B, C . Such a category $\mathcal{C}$ is called a *strictly monoidal category*. For the general case the two diagrams (*) are commutative up to a natural isomorphism and such that the coherence conditions of § 1. hold. By the coherence theorem of [5] this implies that all morphisms composed of $\alpha: A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, $\lambda: I \otimes A \cong A$, $\rho: A \otimes I \cong A$, identities and $\otimes$ which formally have common domain and codomain are equal.

Now consider an object $\mathcal{M}$ in $(\text{Cat}, \times, \mathbf{1})$ on which a strict monoidal category $\mathcal{C}$ operates in the right. Then $\mathcal{M}$ is a category together with a functor $\otimes: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$, such that

$$M \otimes (C \otimes D) = (M \otimes C) \otimes D \quad \text{for all } M \in \mathcal{M}, C, D \in \mathcal{C}$$

and

$$M \otimes I = M \quad \text{for all } M \in \mathcal{M}.$$

These identities are natural transformations in M, C, D . Such a category $\mathcal{M}$ will be called a *strict $\mathcal{C}$ -category*. A useful generalization of this is a $\mathcal{C}$ -category $\mathcal{M}$ for an arbitrary monoidal category $\mathcal{C}$. This is a category $\mathcal{M}$ together with a functor $\otimes: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$ and natural isomorphisms $\beta: M \otimes (C \otimes D) \cong (M \otimes C) \otimes D$ and $\sigma: M \otimes I \cong M$ such that all formal diagrams composed of $\alpha, \beta, \sigma, \lambda, \rho$, their inverses, $\otimes$ in $\mathcal{C}$, and $\otimes$ with respect to $\mathcal{M}$, identities, and compositions commute.

In particular we require the commutativity of the following diagrams:

$$\begin{array}{ccc}
 M \otimes (C \otimes (D \otimes E)) & \xrightarrow{\beta} & (M \otimes C) \otimes (D \otimes E) \\
 \downarrow M \otimes \alpha & & \downarrow \beta \\
 M \otimes ((C \otimes D) \otimes E) & \xrightarrow{\beta} (M \otimes (C \otimes D)) \otimes E \xrightarrow{\beta \otimes E} & ((M \otimes C) \otimes D) \otimes E
 \end{array}$$

$$\begin{array}{ccc}
 M \otimes (I \otimes C) & \xrightarrow{\beta} & (M \otimes I) \otimes C \\
 \searrow M \otimes \lambda & & \swarrow \beta \otimes C \\
 & M \otimes C &
 \end{array}$$

$$\begin{array}{ccc}
 M \otimes (C \otimes I) & \xrightarrow{\beta} & (M \otimes C) \otimes I \\
 \searrow M \otimes \gamma & & \swarrow \beta \\
 & M \otimes C &
 \end{array}$$

If $\mathcal{C}$ is a symmetric monoidal category (which corresponds to the notion of a commutative monoid in $\mathbf{Cat}$), then we also require the commutativity of

$$\begin{array}{ccc}
 M \otimes (I \otimes I) & \xrightarrow{M \otimes \tau} & M \otimes (I \otimes I) \\
 \searrow \beta & & \downarrow \beta \\
 & & (M \otimes I) \otimes I
 \end{array}$$

Now let us regard the corresponding morphisms. A morphism of monoids in $\mathbf{Cat}$ is a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ of strictly monoidal categories $\mathcal{C}$ and $\mathcal{D}$, such that

$$\mathcal{F}(C \otimes D) = \mathcal{F}(C) \otimes \mathcal{F}(D)$$

and

$$\mathcal{F}(I) = J$$

where $I \in \mathcal{C}$ and $J \in \mathcal{D}$ are the neutral objects. The identities are natural transformations in $\mathcal{C}$ and $\mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are just monoidal categories then we require natural isomorphisms

$$\delta: \mathcal{F}(C \otimes D) \cong \mathcal{F}(C) \otimes \mathcal{F}(D)$$

$$\zeta: \mathcal{F}(I) \cong J$$

such that the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{F}(C \otimes I) & \xrightarrow{\delta} & \mathcal{F}(C) \otimes \mathcal{F}(I) & \xrightarrow{\mathcal{F}(C) \otimes \zeta} & \mathcal{F}(C) \otimes J \\ & \searrow \mathcal{F}(\varrho) & & & \downarrow \varrho \\ & & \mathcal{F}(C) & & \\ \mathcal{F}(I \otimes C) & \xrightarrow{\delta} & \mathcal{F}(I) \otimes \mathcal{F}(C) & \xrightarrow{\zeta \otimes \mathcal{F}(C)} & J \otimes \mathcal{F}(C) \\ & \searrow \mathcal{F}(\lambda) & & & \downarrow \lambda \\ & & \mathcal{F}(C) & & \end{array}$$

$$\begin{array}{ccc} \mathcal{F}(C \otimes (D \otimes E)) & \xrightarrow{\delta} & \mathcal{F}(C) \otimes \mathcal{F}(D \otimes E) \xrightarrow{\mathcal{F}(C) \otimes \delta} \mathcal{F}(C) \otimes (\mathcal{F}(D) \otimes \mathcal{F}(E)) \\ \downarrow \mathcal{F}(\alpha) & & \downarrow \alpha \\ \mathcal{F}((C \otimes D) \otimes E) & \xrightarrow{\delta} & \mathcal{F}(C \otimes D) \otimes \mathcal{F}(E) \xrightarrow{\delta \otimes \mathcal{F}(E)} (\mathcal{F}(C) \otimes \mathcal{F}(D)) \otimes \mathcal{F}(E) \end{array}$$

Such a functor is called a *monoidal functor*. If $\mathcal{C}$ and $\mathcal{D}$ are symmetric we require in addition the commutativity of

$$\begin{array}{ccc} \mathcal{F}(C \otimes D) & \xrightarrow{\mathcal{F}(\gamma)} & \mathcal{F}(D \otimes C) \\ \downarrow \delta & & \downarrow \delta \\ \mathcal{F}(C) \otimes \mathcal{F}(D) & \xrightarrow{\gamma} & \mathcal{F}(D) \otimes \mathcal{F}(C). \end{array}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be right $\mathcal{C}$ -objects in $\mathbf{Cat}$ for a strictly monoidal category $\mathcal{C}$. A $\mathcal{C}$ -morphism $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{N}$ is a functor such that

$$\mathcal{F}(M \otimes C) = \mathcal{F}(M) \otimes C,$$

where the identity is a natural transformation in $\mathcal{M}$ and $\mathcal{C}$. In the general case of $\mathcal{C}$ -categories $\mathcal{M}$ and $\mathcal{N}$ for a monoidal category $\mathcal{C}$, a $\mathcal{C}$ -functor is a functor $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{N}$ together with a natural isomorphism

$$\xi: \mathcal{F}(M \otimes C) \cong \mathcal{F}(M) \otimes C$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}(M \otimes (C \otimes D)) & \xrightarrow{\xi} & \mathcal{F}(M) \otimes (C \otimes D) \\ \downarrow \mathcal{F}(\beta) & & \downarrow \beta \\ \mathcal{F}((M \otimes C) \otimes D) & \xrightarrow{\xi} \mathcal{F}(M \otimes C) \otimes D \xrightarrow{\xi \otimes D} & (\mathcal{F}(M) \otimes C) \otimes D \end{array}$$

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes I) & \xrightarrow{\xi} & \mathcal{F}(M) \otimes I \\
 \searrow \mathcal{F}(\phi) & & \swarrow \tilde{\phi} \\
 & \mathcal{F}(M) &
 \end{array}$$

Finally we introduce natural transformations between monoidal functors, resp. between $\mathcal{C}$ -functors. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{G}: \mathcal{C} \rightarrow \mathcal{D}$ be monoidal functors. A natural transformation $\chi: \mathcal{F} \rightarrow \mathcal{G}$ is called a *monoidal transformation* if

$$\begin{array}{ccc}
 \mathcal{F}(C \otimes D) & \xrightarrow{\delta} & \mathcal{F}(C) \otimes \mathcal{F}(D) \\
 \downarrow \chi & & \downarrow \chi \otimes \chi \\
 \mathcal{G}(C \otimes D) & \xrightarrow{\delta} & \mathcal{G}(C) \otimes \mathcal{G}(D)
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathcal{F}(I) & \xrightarrow{\xi} & \mathcal{J} \\
 \chi \downarrow & & \uparrow \xi \\
 \mathcal{G}(I) & &
 \end{array}$$

commute.

A natural transformation $\psi: \mathcal{F} \rightarrow \mathcal{G}$ between $\mathcal{C}$ -functors $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{N}$ and $\mathcal{G}: \mathcal{M} \rightarrow \mathcal{N}$ is called a *$\mathcal{C}$ -morphism* if

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes C) & \xrightarrow{\xi} & \mathcal{F}(M) \otimes C \\
 \downarrow \psi & & \downarrow \psi \otimes C \\
 \mathcal{G}(M \otimes C) & \xrightarrow{\xi} & \mathcal{G}(M) \otimes C
 \end{array}$$

commutes.

Now we can introduce the notion of a right A -object in a right $\mathcal{C}$ -category $\mathcal{D}$, where A is a monoid in $\mathcal{C}$. An object $M \in \mathcal{D}$ together with $v_M: M \otimes A \rightarrow M$ is an *A -object* if the diagrams

$$\begin{array}{ccc}
 M \otimes (A \otimes A) & \xrightarrow{\beta} & (M \otimes A) \otimes A \xrightarrow{v_M \otimes A} M \otimes A \\
 \downarrow M \otimes \mu_A & & \downarrow v_M \\
 M \otimes A & \xrightarrow{v_M} & M
 \end{array}$$

and

$$\begin{array}{ccc}
 M \xrightarrow{\cong} M \otimes I & \xrightarrow{\text{Mon}} & M \otimes A \\
 \searrow \text{id}_M & & \downarrow v_M \\
 & & M
 \end{array}$$

commute. It is clear how A -morphisms are to be defined. Thus we get a category of A -objects $\mathcal{D}_A$.

4.1. Lemma *Let $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{D}'$ be a $\mathcal{C}$ -functor. Then $\mathcal{F}$ induces a functor $\mathcal{F}_A: \mathcal{D}_A \rightarrow \mathcal{D}'_A$.*

PROOF. Let $(M, v_M) \in \mathcal{D}_A$. We define $\mathcal{F}_A(M, v_M) := (\mathcal{F}(M), \mathcal{F}'(v_M))$ where $\mathcal{F}'(v_M)$ is the morphism $\mathcal{F}(M) \otimes A \cong \mathcal{F}(M \otimes A) \xrightarrow{\mathcal{F}(v_M)} \mathcal{F}(M)$. The diagrams

$$\begin{array}{ccccc}
 \mathcal{F}(M) \otimes (A \otimes A) & \cong & (\mathcal{F}(M) \otimes A) \otimes A & \xrightarrow{\mathcal{F}'(v_M) \otimes A} & \mathcal{F}(M) \otimes A \\
 \parallel & & \parallel & \nearrow \mathcal{F}(v_M) \otimes A & \parallel \\
 \mathcal{F}(M \otimes A) \otimes A & & \mathcal{F}(M \otimes A) \otimes A & & \mathcal{F}(M \otimes A) \\
 \parallel & & \parallel & \searrow \mathcal{F}(v_M \otimes A) & \parallel \\
 \mathcal{F}(M \otimes (A \otimes A)) & \cong & \mathcal{F}(M \otimes A) \otimes A & \xrightarrow{\mathcal{F}(v_M \otimes A)} & \mathcal{F}(M \otimes A) \\
 \downarrow \mathcal{F}(M \otimes \mu_A) & & & & \downarrow \mathcal{F}(v_M) \\
 \mathcal{F}(M \otimes A) & \xrightarrow{\mathcal{F}(v_M)} & & & \mathcal{F}(M)
 \end{array}$$

and

$$\begin{array}{ccccc}
 \mathcal{F}(M) \otimes I & \xrightarrow{\mathcal{F}(M) \otimes \eta} & \mathcal{F}(M) \otimes A \\
 \cong \parallel & & \parallel \\
 \mathcal{F}(M) \cong \mathcal{F}(M \otimes I) & \xrightarrow{\mathcal{F}(M \otimes \eta)} & \mathcal{F}(M \otimes A) \\
 \searrow \text{id}_{\mathcal{F}(M)} & & \downarrow \mathcal{F}(v_M) \\
 & & \mathcal{F}(M)
 \end{array}$$

commute. Similarly if $f: M \rightarrow N$ is a morphism in $\mathcal{D}_A$ then $\mathcal{F}(f)$ is in $\mathcal{D}'_A$ since

$$\begin{array}{ccc}
 \mathcal{F}(M) \otimes A & \xrightarrow{\mathcal{F}(f) \otimes A} & \mathcal{F}(N) \otimes A \\
 \parallel & & \parallel \\
 \mathcal{F}(M \otimes A) & \xrightarrow{\mathcal{F}(f \otimes A)} & \mathcal{F}(N \otimes A) \\
 \mathcal{F}(v_M) \downarrow & & \downarrow \mathcal{F}(v_N) \\
 \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(N)
 \end{array}$$

commutes. So define $\mathcal{F}_A(f) := \mathcal{F}(f)$ in $\mathcal{D}'_A$ and we get a functor $\mathcal{F}_A: \mathcal{D}_A \rightarrow \mathcal{D}'_A$.

This proof shows already why we had to require certain coherence conditions for the definition of $\mathcal{C}$ -functors. The most important property of $\mathcal{C}$ -functors which is very similar to properties of exact functors will be discussed later on. First we need a few additional properties of A -objects in $\mathcal{C}$.

By Theorem 2.2 and the remark following 2.3 Lemma 3 of [12] the following is a difference cokernel of a contractible pair in $\mathcal{C}$

$$A \otimes (A \otimes M) \xrightarrow[A \otimes v_M]{(\mu_A \otimes M) \circ \alpha} A \otimes M \xrightarrow{v_M} M$$

for each A -object M in $\mathcal{C}$. In fact only the contraction morphism $(\eta \otimes (A \otimes M)) \cdot \lambda^{-1}: A \otimes M \rightarrow A \otimes (A \otimes M)$ is in $\mathcal{C}$, the morphisms of the above sequence are even in ${}_A\mathcal{C}$. So we have a difference cokernel of a $(\mathcal{U}: {}_A\mathcal{C} \rightarrow \mathcal{C})$ -contractible pair in ${}_A\mathcal{C}$.

- 4.2. **Theorem.** Let $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ be a covariant functor. Equivalent are
- $\mathcal{F}$ is a $\mathcal{C}$ -functor and $\mathcal{F}$ preserves difference cokernels of $\mathcal{U}$ -contractible pairs.
 - There is a $B-A$ -biobject Q which is A -coflat (which is unique up to an isomorphism) such that $\mathcal{F} \cong Q \otimes_A$ as functors from ${}_A\mathcal{C}$ to ${}_B\mathcal{C}$.

PROOF. Assume that a) holds. Then we have a difference cokernel

$$\mathcal{F}(A \otimes (A \otimes M)) \rightrightarrows \mathcal{F}(A \otimes M) \rightarrow \mathcal{F}(M).$$

Since $\mathcal{F}$ is a $\mathcal{C}$ -functor we get a difference cokernel

$$\mathcal{F}(A) \otimes (A \otimes M) \xrightarrow[\mathcal{F}(A) \otimes v_M]{(\mathcal{F}'(\mu_A) \otimes M) \circ \alpha} \mathcal{F}(A) \otimes M \xrightarrow{\mathcal{F}(v_M) \circ \xi^{-1}} \mathcal{F}(M).$$

By definition of $\mathcal{F}(A) \otimes_A M$ in §3 we get a natural isomorphism $\mathcal{F}(M) \cong \mathcal{F}(A) \otimes_A M$ with the $B-A$ -biobject $\mathcal{F}(A) = Q$. If $Q' \otimes_A \cong \mathcal{F}$ then $Q' \otimes_A A \cong Q \otimes_A A$ hence $Q' \cong Q$ as $B-A$ -biobjects.

Conversely assume that b) holds. The following commutative diagram indicates that $Q \otimes_A: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ is a $\mathcal{C}$ -functor:

$$\begin{array}{ccccc} (Q \otimes (A \otimes M)) \otimes X & = & (Q \otimes M) \otimes X & \rightarrow & (Q \otimes_A M) \otimes X \\ \parallel & & \parallel & & \parallel \\ Q \otimes (A \otimes (M \otimes X)) & = & Q \otimes (M \otimes X) & \rightarrow & Q \otimes_A (M \otimes X). \end{array}$$

The verification of the coherence diagrams for $\mathcal{C}$ -functors is left to the reader. Now let

$$M \xrightleftharpoons[f_1]{f_0} N \xrightarrow{h} P$$

be a difference cokernel of a $\mathcal{U}$ -contractible pair in ${}_A\mathcal{C}$ with contraction $g: N \rightarrow M$ in $\mathcal{C}$ and section $k: P \rightarrow N$ in $\mathcal{C}$ such that $f_0 g = id_N$, $f_1 g f_0 = f_1 g f_1$, $h k = id_P$ and $h f_1 g = f_1 g$. Now $Q \otimes$ preserves the whole situation, so we get a commutative diagram

$$\begin{array}{ccccc} Q \otimes N & \xrightarrow{id_{Q \otimes N}} & & & Q \otimes N \\ & \searrow Q \otimes g & & \nearrow Q \otimes f_0 & \\ & & Q \otimes M & & \\ & & \downarrow Q \otimes f_1 & & \\ & & Q \otimes N & & \\ Q \otimes P & \xleftarrow{Q \otimes k} & & \xrightarrow{Q \otimes h} & Q \otimes P \\ & \searrow id_{Q \otimes P} & & \nearrow & \end{array}$$

This implies that

$$Q \otimes M \xrightarrow[Q \otimes f_1]{Q \otimes f_0} Q \otimes N \xrightarrow{Q \otimes h} Q \otimes P$$

is a difference cokernel of a $\mathcal{U}$ -contractible pair [12, 2.3].

A similar diagram is obtained by tensoring with $Q \otimes A$ on the left. This induces a commutative diagram.

$$\begin{array}{ccccc} (Q \otimes A) \otimes M & \Rightarrow & (Q \otimes A) \otimes N & \rightarrow & (Q \otimes A) \otimes P \\ \Downarrow & & \Downarrow & & \Downarrow \\ Q \otimes M & \Rightarrow & Q \otimes N & \rightarrow & Q \otimes P \\ \downarrow & & \downarrow & & \downarrow \\ Q \otimes_A M & \Rightarrow & Q \otimes_A N & \rightarrow & Q \otimes_A P \end{array}$$

where all rows and columns are difference cokernels in ${}_B\mathcal{C}$, due to the fact that f_0, f_1 and h are A -morphisms and that colimits commute with colimits. Hence $Q \otimes_A : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ preserves difference cokernels of $\mathcal{U}$ -contractible pairs.

It seems that the strong property of being a $\mathcal{C}$ -functor rarely occurs. But the following theorem shows that this property is closely related to inner hom functors. Let us first consider the case $\mathcal{C} = \mathbf{Z}\text{-Mod}$. Let $\mathcal{F} : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ have a right adjoint $\mathcal{G} : {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$. Then $\mathcal{F}$ and $\mathcal{G}$ preserve colimits resp. limits hence they are additive functors. Now the natural bijection

$${}_B\mathcal{C}(\mathcal{F}(M), N) \cong {}_A\mathcal{C}(M, \mathcal{G}(N)) \quad \text{is composed of}$$

$$\mathcal{G} : {}_B\mathcal{C}(\mathcal{F}(M), N) \rightarrow {}_A\mathcal{C}(\mathcal{G}\mathcal{F}(M), \mathcal{G}(N))$$

and

$${}_A\mathcal{C}(\Phi M, \mathcal{G}(N)) : {}_A\mathcal{C}(\mathcal{G}\mathcal{F}(M), \mathcal{G}(N)) \rightarrow {}_A\mathcal{C}(M, \mathcal{G}(N)).$$

Both maps are homomorphisms of abelian groups since $\mathcal{G}$ is an additive functor and $\Phi M : M \rightarrow \mathcal{G}\mathcal{F}(M)$ is in ${}_A\mathcal{C}$. Thus a pair of adjoint functors between ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ with $\mathcal{C} = \mathbf{Z}\text{-Mod}$ is automatically such that not only the morphism sets ${}_B\mathcal{C}(\mathcal{F}(M), N)$ and ${}_A\mathcal{C}(M, \mathcal{G}(N))$ are isomorphic in the category of sets but even the inner hom functors ${}_B[\mathcal{F}(M), N]$ and ${}_A[M, \mathcal{G}(N)]$ are isomorphic in $\mathcal{C} = \mathbf{Z}\text{-Mod}$. Unfortunately this is not true in the general case. However the following theorem holds

4.3. Theorem. a) Let $\mathcal{C}$ be a closed monoidal category and $\mathcal{F} : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G} : {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ be functors. $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and a $\mathcal{C}$ -functor if and only if there is a natural isomorphism

$${}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)]$$

for all $M \in {}_A\mathcal{C}$ and $N \in {}_B\mathcal{C}$.

b) Let $\mathcal{C}$ be a coclosed monoidal category and $\mathcal{F} : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G} : {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ be functors. $\mathcal{F}$ is right adjoint to $\mathcal{G}$ and a $\mathcal{C}$ -functor if and only if there is a natural isomorphism

$${}_B\langle \mathcal{F}(M), N \rangle \cong {}_A\langle M, \mathcal{G}(N) \rangle$$

for all $M \in {}_A\mathcal{C}$ and $N \in {}_B\mathcal{C}$.

PROOF. First assume that there is a natural isomorphism

$$\Phi: {}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)].$$

Then $\mathcal{F}$ is left adjoint to $\mathcal{G}$ by ${}_B\mathcal{C}(\mathcal{F}(M), N) \cong {}_B\mathcal{C}(\mathcal{F}(M) \otimes I, N) \cong {}_B\mathcal{C}(I, {}_B[\mathcal{F}(M), N]) \cong {}_B\mathcal{C}(I, {}_A[M, \mathcal{G}(N)]) \cong {}_A\mathcal{C}(M \otimes I, \mathcal{G}(N)) \cong {}_A\mathcal{C}(M, \mathcal{G}(N))$. Call this isomorphism φ . It is clear that φ is natural in M and N .

Now define $\xi: \mathcal{F}(M \otimes C) \cong \mathcal{F}(M) \otimes C$ by ${}_B\mathcal{C}(\xi, N)$ as ${}_B\mathcal{C}(\mathcal{F}(M) \otimes C, N) \cong {}_B\mathcal{C}(C, {}_B[\mathcal{F}(M), N]) \cong {}_B\mathcal{C}(C, {}_A[M, \mathcal{G}(N)]) \cong {}_A\mathcal{C}(M \otimes C, \mathcal{G}(N)) \cong {}_B\mathcal{C}(\mathcal{F}(M \otimes C), N)$. Again ξ is clearly a natural transformation in M and C .

To check the two properties for a $\mathcal{C}$ -functor denote by

$$\psi_A: {}_A\mathcal{C}(M \otimes C, M') \cong {}_A\mathcal{C}(C, {}_A[M, N'])$$

$$\psi_B: {}_B\mathcal{C}(N \otimes C, N') \cong {}_B\mathcal{C}(C, {}_B[N, N'])$$

the adjointness isomorphisms of 3.10. Then the diagram

$$\begin{array}{ccccc} {}_B\mathcal{C}(\mathcal{F}(M), N) & \xrightarrow{\varphi} & {}_A\mathcal{C}(M, \mathcal{G}(N)) & & \\ \downarrow {}_B\mathcal{C}(\xi, N) & \searrow {}_B\mathcal{C}(\mathcal{F}(\delta), N) & \downarrow {}_A\mathcal{C}(\delta, \mathcal{G}(N)) & & \\ {}_B\mathcal{C}(\mathcal{F}(M) \otimes I, N) & \xrightarrow{{}_B\mathcal{C}(\xi, N)} & {}_B\mathcal{C}(\mathcal{F}(M \otimes I), N) & \xrightarrow{\varphi} & {}_A\mathcal{C}(M \otimes I, \mathcal{G}(N)) \\ \downarrow \psi_B & & & & \downarrow \psi_A \\ {}_A\mathcal{C}(I, {}_B[\mathcal{F}(M), N]) & \xrightarrow{{}_A\mathcal{C}(I, \Phi)} & {}_A\mathcal{C}(I, {}_A[M, \mathcal{G}(N)]) & & \end{array}$$

commutes, the outer hexagon by definition of φ , the lower pentagon by definition of ξ , the upper right hand quadrangle since φ is a natural transformation and the upper left hand triangle since all morphisms of the diagram are isomorphisms. Hence we get a commutative diagram in ${}_B\mathcal{C}$

$$\begin{array}{ccc} \mathcal{F}(M \otimes I) & \xrightarrow{\xi} & \mathcal{F}(M) \otimes I \\ & \searrow \mathcal{F}(\delta) & \swarrow \delta \\ & \mathcal{F}(M) & \end{array}$$

For the second more complicated diagram for $\mathcal{C}$ -functors observe first from § 3 that we have natural isomorphisms

$${}_A[M \otimes C, M'] \cong [C, {}_A[M, M']] \quad \text{and} \quad {}_B[N \otimes C, N'] \cong [C, {}_B[N, N']]$$

and that the diagram

$$\begin{array}{ccc} [C \otimes D, {}_A[M, M']] & \cong & {}_A[M \otimes (C \otimes D), M'] \cong {}_A[(M \otimes C) \otimes D, M'] \\ & \cong & \\ [D, [C, {}_A[M, M']]] & \cong & [D, {}_A[M \otimes C, M']] \end{array}$$

and the corresponding diagram with respect to ${}_B\mathcal{C}$ commute. Thus we get a commutative diagram

$$\begin{array}{ccc}
 {}_A[M \otimes (C \otimes D), \mathcal{G}(N)] & \cong & {}_A[(M \otimes C) \otimes D, \mathcal{G}(N)] \cong [D, {}_A[M \otimes C, \mathcal{G}(N)]] \\
 \parallel & & \parallel \\
 [C \otimes D, {}_A[M, \mathcal{G}(N)]] & \xrightarrow{\quad} & [D, [C, {}_A[M, \mathcal{G}(N)]]] \\
 \parallel & & \parallel \\
 [C \otimes D, {}_B[\mathcal{F}(M), N]] & \xrightarrow{\quad} & [D, [C, {}_B[\mathcal{F}(M), N]]] \\
 \parallel & & \parallel \\
 {}_B[\mathcal{F}(M) \otimes (C \otimes D), N] & \cong & {}_B[(\mathcal{F}(M) \otimes C) \otimes D, N] \cong [D, {}_B[\mathcal{F}(M) \otimes C, N]].
 \end{array}$$

Observe now the following diagram

$$\begin{array}{ccccc}
 {}_A[M \otimes (C \otimes D), \mathcal{G}(N)] & \xrightarrow{\quad} & [C \otimes D, {}_A[M, \mathcal{G}(N)]] & \xrightarrow{\quad} & [C \otimes D, {}_B[\mathcal{F}(M), N]] \\
 \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
 {}_B[\mathcal{F}(M \otimes (C \otimes D)), N] & \xleftarrow{{}_B[\xi, N]} & {}_B[\mathcal{F}(M) \otimes (C \otimes D), N] & & \uparrow \\
 & & \uparrow & & \uparrow \\
 & & {}_B[\mathcal{F}(\beta), N] & & \uparrow \\
 & & \uparrow & & \uparrow \\
 {}_B[\mathcal{F}((M \otimes C) \otimes D), N] & \xleftarrow{{}_B[\xi, N]} & {}_B[\mathcal{F}(M \otimes C) \otimes D, N] & \xleftarrow{{}_B[\xi \otimes D, N]} & {}_B[\mathcal{F}(M) \otimes C \otimes D, N] \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 {}_A[(M \otimes C) \otimes D, \mathcal{G}(N)] & \xrightarrow{\quad} & [D, {}_B[\mathcal{F}(M \otimes C), N]] & \xleftarrow{\quad} & [D, {}_B[\mathcal{F}(M) \otimes C, N]] \\
 & \searrow & \downarrow & \searrow & \downarrow \\
 & & [D, {}_A[M \otimes C, \mathcal{G}(N)]] & \xrightarrow{\quad} & [D, [C, {}_B[\mathcal{F}(M), N]]] \\
 & & \searrow & & \downarrow \\
 & & & & [D, [C, {}_A[M, \mathcal{G}(N)]]]
 \end{array}$$

The outer frame commutes, since the previous diagram commutes.

The left quadrangle is commutative since Φ is a natural transformation, the right hand square since ψ_B is a natural transformation, the three outer pentagons by the definition of ξ . Since all morphisms are isomorphisms, the inner pentagon commutes also. Hence we get

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes (C \otimes D)) & \xrightarrow{\quad} & \mathcal{F}(M) \otimes (C \otimes D) \\
 \downarrow \mathcal{F}(\beta) & & \downarrow \beta \\
 \mathcal{F}((M \otimes C) \otimes D) & \xrightarrow{\xi} \mathcal{F}(M \otimes C) \otimes D \xrightarrow{\xi \otimes D} & (\mathcal{F}(M) \otimes C) \otimes D
 \end{array}$$

commutative. This proves one direction of part a) of the theorem.

Conversely, let $\mathcal{F}$ be a $\mathcal{C}$ -functor and left adjoint to $\mathcal{G}$. Then we have isomorphisms

$$\begin{aligned}
 \mathcal{C}(C, {}_B[\mathcal{F}(M), N]) &\cong {}_B\mathcal{C}(\mathcal{F}(M) \otimes C, N) \cong {}_B\mathcal{C}(\mathcal{F}(M \otimes C), N) \cong \\
 &{}_A\mathcal{C}(M \otimes C, \mathcal{G}(N)) \cong \mathcal{C}(C, {}_A[M, \mathcal{G}(N)])
 \end{aligned}$$

natural in $C \in \mathcal{C}$, $M \in {}_A\mathcal{C}$ and $N \in {}_B\mathcal{C}$. Hence ${}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)]$ is a natural isomorphism.

The proof of part b) of the theorem is essentially dual to the proof of part a) and is left to the reader.

4.4. Corollary: Let $\mathcal{F} : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G} : {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ be functors such that there is a natural isomorphism ${}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)]$. Then there is an A -coflat object $Q \in {}_B\mathcal{C}_A$ (unique up to an isomorphism) such that

$$\mathcal{F}(M) \cong Q \otimes_A M \quad \text{and} \quad \mathcal{G}(N) \cong {}_B[Q, N]$$

natural in M resp. N .

Proof: Since $\mathcal{F}$ is a $\mathcal{C}$ -functor and preserves colimits, Theorem 4.2. implies $\mathcal{F}(M) \cong Q \otimes_A M$. By Proposition 3.11 we also get $\mathcal{G}(N) \cong {}_B[Q, N]$.

If $\mathcal{C} = \mathbf{Z}\text{-Mod}$ we know in the situation of Theorem 4.3 that $\mathcal{F}$ is an additive functor. This holds more generally.

4.5. Proposition: Let $\mathcal{C} = K\text{-Mod}$ with the usual tensor-product. Let $\mathcal{F} : {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ be a $\mathcal{C}$ -functor. Then $\mathcal{F}$ is K -additive, i.e. for all $f, g \in {}_A\mathcal{C}(M, N)$, $\kappa \in K$ we have

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g) \quad \text{and} \quad \mathcal{F}(\kappa f) = \kappa \mathcal{F}(f).$$

PROOF. First show that there is a natural isomorphism $\mathcal{F}(M \oplus M) \cong \mathcal{F}(M) \oplus \mathcal{F}(M)$ (natural in $M \in {}_A\mathcal{C}$) such that

$$(*) \quad \begin{array}{ccc} & \mathcal{F}(M \oplus M) & \\ \mathcal{F}(q_i) \nearrow & \cong & \nwarrow \mathcal{F}(p_i) \\ \mathcal{F}(M) & & \mathcal{F}(M) \\ & \mathcal{F}(M) \oplus \mathcal{F}(M) & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathcal{F}(M \oplus M) & \\ \mathcal{F}(M \oplus M) \nearrow & \cong & \nwarrow \mathcal{F}(M \oplus M) \\ \mathcal{F}(M) \oplus \mathcal{F}(M) & & \mathcal{F}(M) \oplus \mathcal{F}(M) \end{array}$$

for $i=1, 2$ commute, where q_i are the i -th injections and p_i are the i -th projections of the direct sum. The isomorphism is given by $\mathcal{F}(M \oplus M) \cong \mathcal{F}(M \otimes (K \oplus K)) \cong \mathcal{F}(M) \otimes (K \oplus K) \cong \mathcal{F}(M) \oplus \mathcal{F}(M)$. It is clearly natural in $M \in {}_A\mathcal{C}$. The commutativity of the first diagram follows from

$$\begin{array}{ccccccc} \mathcal{F}(M) & \xrightarrow{\quad id_{\mathcal{F}(M)} \quad} & & & \mathcal{F}(M) & & \\ \downarrow \mathcal{F}(q_i) & \cong & \mathcal{F}(M \otimes K) & \cong & \mathcal{F}(M) \otimes K & \cong & \downarrow q_i \\ \mathcal{F}(M \oplus M) & \cong & \mathcal{F}(M \otimes (K \oplus K)) & \cong & \mathcal{F}(M) \otimes (K \oplus K) & \cong & \mathcal{F}(M) \oplus \mathcal{F}(M) \end{array}$$

The other diagram follows similarly.

The diagrams (*) imply immediately the commutativity of

$$\begin{array}{ccc} & \mathcal{F}(M) & \\ \mathcal{F}(q_i) \nearrow & \Delta & \nwarrow \mathcal{F}(p_i) \\ \mathcal{F}(M \oplus M) & \cong & \mathcal{F}(M) \oplus \mathcal{F}(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathcal{F}(M \oplus M) & \\ \mathcal{F}(M \oplus M) \nearrow & \Delta & \nwarrow \mathcal{F}(M \oplus M) \\ \mathcal{F}(M) \oplus \mathcal{F}(M) & & \mathcal{F}(M) \oplus \mathcal{F}(M) \end{array}$$

Furthermore they imply the commutativity of

$$\begin{array}{ccc} \mathcal{F}(M \oplus M) & \xrightarrow{\mathcal{F}(f \oplus g)} & \mathcal{F}(N \oplus N) \\ \parallel & & \parallel \\ \mathcal{F}(M) \oplus \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f) \oplus \mathcal{F}(g)} & \mathcal{F}(N) \oplus \mathcal{F}(N). \end{array}$$

Hence

$$\begin{array}{ccccc} \mathcal{F}(M) & \xrightarrow{\quad} & & \xrightarrow{\quad} & \mathcal{F}(N) \\ & \searrow \mathcal{F}(\Delta) & & \nearrow \mathcal{F}(\nabla) & \\ \Delta \downarrow & \mathcal{F}(M \oplus M) & \xrightarrow{\mathcal{F}(f \oplus g)} & \mathcal{F}(N \oplus N) & \uparrow \nabla \\ \mathcal{F}(M) \oplus \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f) \oplus \mathcal{F}(g)} & & \mathcal{F}(N) \oplus \mathcal{F}(N) \end{array}$$

commutes, where the first horizontal arrow is $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$.

To show that $\mathcal{F}$ is K -linear, observe that for $f \in {}_A\mathcal{C}(M, N)$ and $\kappa \in K$ the diagram

$$\begin{array}{ccc} M \otimes K & \xrightarrow{f \otimes \kappa} & N \otimes K \\ \parallel & & \parallel \\ M & \xrightarrow{\kappa} & N \end{array}$$

commutes. Hence the diagram

$$\begin{array}{ccccc} \mathcal{F}(M) \otimes K & \xrightarrow{\mathcal{F}(f) \otimes \kappa} & & \xrightarrow{\quad} & \mathcal{F}(N) \otimes K \\ \parallel & \nearrow & \mathcal{F}(M \otimes K) & \xrightarrow{\mathcal{F}(f \otimes \kappa)} & \mathcal{F}(N \otimes K) \\ \mathcal{F}(M) & \xrightarrow{\quad} & & \xrightarrow{\quad} & \mathcal{F}(N) \end{array}$$

commutes, where the lower horizontal arrow is $\mathcal{F}(\kappa f) = \kappa \mathcal{F}(f)$.

Bibliography

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