

The archimedean kernel of a lattice-ordered group

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0. *Terminology and notation.* Throughout this paper G will denote a lattice-ordered group (henceforth, l -group), written additively without regard to the commutativity of G . A subgroup H of G is called an l -subgroup if it is also a sublattice and is convex if $x \in H$ and $0 < g \leq x$ implies $g \in H$. If H is an l -subgroup of G , the positive cone of H , denoted H^+ , is the set $\{h \in H \mid 0 \leq h\}$. The polar of a subset $A \subseteq G$, denoted A' , is the set $\{x \in G \mid |x| \wedge |a| = 0 \text{ for all } a \in A\}$ and we let $\{a'\} = a'$. A value for $0 \neq x \in G$, denoted P_x , is a maximal element of the set of all convex l -subgroups of G which do not contain x . Let $\Gamma(G)$ be an index set for the set of all values of non-zero elements of G partially ordered by $\gamma \leq \delta$ if $G_\gamma \subseteq G_\delta$. The intersection of all convex l -subgroups of G which properly contain a given value G_γ is the smallest convex l -subgroup properly containing G_γ and is called the cover of G_γ . The cover of G_γ is denoted G^γ . G is called normal-valued if each value is a normal subgroup of its cover. For $x \in G$, $G(x)$ will denote the convex l -subgroup generated by x . For other terminology and notation, the reader is referred to [7] or [9].

1. *The existence of the archimedean kernel.* Our original proof of the existence of the archimedean kernel required that G be representable ([10]). If $G(g)$ is archimedean, $G(g)$ is contained in the largest abelian convex l -subgroup of G , so the existence of the archimedean kernel for representable l -groups implies the existence in general. In [13], REDFIELD proved the existence of the archimedean kernel for an arbitrary l -group. Since Redfield's proof is easier, we will use it here.

An element $a \in G^+$ is called archimedean if for each $0 < g \leq a$, there exists an integer $n > 0$ such that $ng \not\leq a$. Let $P(G)$ be the set of all archimedean elements of G^+ and let $A(G)$ be the convex l -subgroup generated by $P(G)$.

Theorem 1.1. (REDFIELD, [13]). $A(G)^+ = P(G)$.

PROOF. Since $P(G)$ is a convex normal subset of G which contains 0, it suffices to show $P(G)$ is a subsemigroup of G^+ . Suppose (by way of contradiction) that there exists $a, b \in P(G)$ such that $a + b \notin P(G)$. Then there is a $0 < t \leq a + b$ such that $nt \leq a + b$ for all integers $n > 0$. Since a is archimedean, there exists an integer $m > 0$ such that $mt \not\leq a$ ($m = 1$ is possible). Then

$$s = (-a + mt) \vee 0 > 0.$$

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Since $nt \leq a+b$ for all integers $n > 0$,

$$(1) \quad -a + nt \leq b \quad \text{for all } n > 0,$$

so

$$(2) \quad (-a + nt) \vee 0 \leq b \quad \text{for all } n > 0.$$

We will show by induction on k that

$$ks \leq (-a + kmt) \vee 0 \quad \text{for all } k > 0.$$

The case when $k=1$ is valid by the definition of s . Suppose that the inequality is valid for $k=h$, i.e.,

$$(3) \quad hs \leq (-a + hmt) \vee 0.$$

Then,

$$(h+1)s = hs + s \leq (-a + hmt) \vee 0 + (-a + mt) \vee 0$$

by (3). Since $-a + hmt \leq -a + mt$ and $-a + hmt + mt \leq -a + hmt - a + mt$,

$$\begin{aligned} (h+1)s &\leq (-a + hmt - a + mt) \vee (-a + hmt) \vee (-a + mt) \vee 0 \leq \\ &\leq (-a + hmt + mt) \vee (-a + hmt) \vee 0. \end{aligned}$$

Therefore,

$$(h+1)s \leq (-a + (h+1)mt) \vee (-a + mt) \vee 0 \leq (-a + (h+1)mt) \vee 0,$$

which completes the induction. Now, by (2),

$$ks \leq (-a + kmt) \vee 0 \leq b$$

for all $k > 0$ which is impossible since b is archimedean. Therefore, $a+b$ is archimedean and so $S(G) = A(G)^+$.

Corollary 1.2. (REDFIELD, [13]) $A(G)$ is the (unique) largest convex archimedean l -subgroup of G .

An element $0 \neq s \in G^+$ is called *basic* if the set $\{x \in G^+ \mid x \leq s\}$ is totally ordered. G has a *basis* if each positive element exceeds a basic element. If $s \in G$ is basic, then s' is a prime convex l -subgroup of G ([17]) so s has a unique value. A convex l -subgroup K of G is said to be *closed* if for each subset $\{h_\lambda \mid \lambda \in B\}$ of K such that $h = \bigvee h_\lambda$ exists in G , then $h \in K$.

Theorem 1.3. Let G be normal-valued and let $0 \neq x \in G$. The following are equivalent:

- (a) $x \in A(G)$.
- (b) $|x|$ is archimedean.
- (c) $G(|x|)$ is an archimedean l -group.
- (d) $x' = \bigcap \{P_x \mid P_x \text{ is a value for } x\}$.

If, in addition, G has radical zero (see [7]) and Δ is the minimal plenary subset of $\Gamma(G)$, each of the above is equivalent to

- (e) If $\delta \in \Delta$ is a value for x , then δ is a minimal element of $\Gamma(G)$.

PROOF. The equivalence of (a), (b), and (c) can be found in [13].

(c)→(d) Clearly $x' \subseteq \cap P_x$. Let $y \in \cap P_x$. Then

$$|x| \wedge |y| \in G(|x|) \cap (\cap P_x) = \cap (P_x \cap G(|x|)) = \{\text{maximal } l\text{-ideals of } G(|x|)\} = 0$$

since $G(|x|)$ is an archimedean l -group with a strong order unit. Thus $y \in x'$ and $x' = \cap P_x$.

(d)→(b) Suppose (by way of contradiction) there exists $0 < t \leq x$ such that $nt \leq x$ for all integers $n > 0$. Since $0 \neq t = t \wedge |x|$, by (d), there is a value P_x of x such that $t \notin P_x$. Since $|x| \geq t > 0$, P_x is also a value for t . Now

$$|x| + P_x \geq nt + P > P_x \text{ for all integers } n > 0$$

which is impossible since G is normal-valued. Thus $|x|$ is archimedean.

Now suppose G has radical zero and Δ is the minimal plenary subset of $\Gamma(G)$.

(a)→(e) Since G has radical zero, so does $A(G)$ and so $A(G)$ has a basis, (see [7]), say $S = \{s_\lambda | \lambda \in \Delta\}$. Then an isomorphic copy of $A(G)$ lies between $\Sigma_A G(s_\lambda)$ and $\Pi_A G(s_\lambda)$. For each $\lambda \in \Delta$, let $G_\lambda = s'_\lambda$ be the unique value of s_λ , (since $s_\lambda \in A(G)$, $s'_\lambda = \cap \{P | P \text{ is a value of } s_\lambda\} = G_\lambda$). Since G_λ is a polar, it is a minimal prime. Let $0 \neq x \in A(G)$ and let $\delta \in \Delta$ be a value for x . Since G_δ is essential (see [7]), it is closed, so there exists a $\lambda \in \Delta$ such that $s_\lambda \notin G_\delta$. Therefore, $G_\delta = G_\lambda$ and δ is a minimal element of $\Gamma(G)$.

(e)→(b) Let $x \in G$ be such that each value for x in Δ is a minimal element of $\Gamma(G)$. Suppose (by way of contradiction) there exists $0 < t \in G$ such that $nt \leq |x|$ for all integers $n > 0$. Let $\delta \in \Delta$ be a value for t . Since $x \notin G$ and each value for x in Δ is minimal, δ is a value for x . But then, $|x| + G_\delta \geq nt + G_\delta > G_\delta$ which is impossible since G is normal-valued. Thus $|x|$ is archimedean.

Remark. The proof of theorem 1.1 shows that (e)→(a)↔(b)↔(c)→(d) is valid for arbitrary l -groups.

2. Properties of $A(G)$. If G is an l -subgroup of H , then H is called an a^* -extension of G if $K \rightarrow K \cap G$ is a one-to-one map of the closed convex l -subgroups of H onto those of G .

G is said to be an

L -group if $\forall s_\lambda$ exists for any disjoint subset $\{s_\lambda\}$ of G .

P -group if $G = g' \oplus g''$ for all $g \in G$.

SP -group if $G = A' \oplus A''$ for all $A \subseteq G$.

O -group if G is an L -group and a P -group.

Let G be an l -subgroup of H . H is called an X -hull of G , for $X = P, SP, L$ or O , if G is large in H , H is an X -group and no proper l -subgroup of H which contains G is an X -group. The X -hull of G will be denoted by G^X . Notice that if $X = P, SP$ or O , then G^X , and hence G , is representable. For discussion of X -hulls, see [8], and for the existence of L -hulls, see [1].

Proposition 2.1.

(a) $A(G)$ is a closed l -characteristic l -subgroup of G .

(b) Let K be an l -subgroup of G . If K is either large or convex in G , then $A(K) = K \cap A(G)$.

(c) If G is an X -group, so is $A(G)$ for $X = P, SP, L$ or O .

(d) $A(G)^X \subseteq A(G^X)$ but equality need not hold for $X=P$ or SP even if G is abelian with a basis.

(e) $A(\Pi G_\lambda) = \Pi A(G_\lambda)$.

PROOF. (a) Clearly $A(G)$ is l -characteristic. BLEIER and CONRAD ([2] and [3]) have shown that the closure of a convex l -subgroup is an a^* -extension and that an a^* -extension of an archimedean l -group is archimedean. Thus, maximality of $A(G)$ implies that it is closed.

(b) If K is convex in G , the result is clear. $A(K) \supseteq A(G) \cap K$ is true for any l -subgroup K of G . Suppose K is large in G and let $0 < x \in A(K)$. Let $g \in G$ be such that $0 < g < x$. Since K is large in G , there is an integer $n > 0$ and a $k \in K$ such that $0 < k < ng$. Since $x \in A(K)$, there is an integer $m > 0$ such that $mk \not\leq x$ ($m=1$ is possible). Then $mng \not\leq x$ so $x \in A(G)$.

(c) By [8], a closed convex l -subgroup of an X -group is an X -group.

(d) Since G is large in G^X , $A(G)$ is large in $A(G^X)$, and by (c), $A(G^X)$ is an X -group. The intersection of all l -subgroups of $A(G^X)$ which contain $A(G)$ and are X -groups is the X -hull of $A(G)$. Thus $A(G)^X \subseteq A(G^X)$. For the last statement, see example 1 in section 4.

(e) Clear.

Let K be an l -subgroup of G . K is said to be an $\mathcal{L}$ -subgroup of G if, for each set $\{u_\beta | \beta \in B\}$ of disjoint elements of K such that $\bigvee_K u_\beta$ exists, it follows that $\bigvee_G u_\beta$ exists and equals $\bigvee_K u_\beta$. An H -representation of G is a pair $(\sigma, \Pi H_\lambda)$ where σ is an l -isomorphism of G onto a subdirect sum of ΠH_λ and each H_λ is a transitive l -subgroup of the l -group of all permutations of a totally ordered set T_λ . An H -representation is called *complete* if σ preserves all joins and intersections existing in G . BYRD and LLOYD, [4], have shown that G is completely distributive if and only if G has a complete H -representation $(\sigma, \Pi H_\lambda)$ and, [5], the intersection of all laterally complete $\mathcal{L}$ -subgroups of ΠH_λ which contain $G\sigma$ is the L -hull of G .

Theorem 2.2. *If G is completely distributive, then $A(G)^L = A(G^L)$.*

PROOF. Let $(\sigma, \Pi H_\lambda)$ be a complete H -representation of G . By abuse of notation, we will suppress σ and view G as an l -subgroup of ΠH_λ . Now, $A(G)$ is completely distributive and so has a basis, say $S = \{s_\beta | \beta \in B\}$. S is also a basis for $A(G^L)$. Let $0 < x \in A(G^L)$. For each $\beta \in B$, there is an integer $n_\beta > 0$ such that $n_\beta s_\beta \not\leq x$. Let $x_\beta = n_\beta s_\beta \wedge x$. Then $x_\beta \wedge x_\gamma = 0$ for $\beta \neq \gamma$ and $x = \bigvee x_\beta$. Thus, in order to show $A(G)^L = A(G^L)$, we need only show $x_\beta \in G$. If $x_\beta = 0$, we are done. Suppose $x_\beta > 0$. Then there is a λ such that $(x_\beta)_\lambda > 0$. Since G is a subdirect sum of ΠH_λ , there is a $0 < g \in G$ such that $g_\lambda = (x_\beta)_\lambda$. By replacing g by $n_\beta s_\beta \wedge g$, we may assume $g \in G(s_\beta)$. Since s_β is basic, $G^L(s_\beta)$ is totally ordered and so a subgroup of the real numbers, since $s_\beta \in A(G^L)$. Since $g_\lambda = (x_\beta)_\lambda$, we have $ng > mx_\beta > 0$ if $n > m > 0$, and $mx_\beta > ng > 0$ if $m > n > 0$. Thus $g = x_\beta$ and $A(G^L) = A(G)^L$.

Since the class of archimedean l -groups is not closed under l -homomorphic images, $A(G)\beta$ need not be contained in $A(G\beta)$ for an arbitrary l -homomorphism β . It is natural to ask when $A(G)\beta$ is contained in $A(G\beta)$ for all l -homomorphisms β . If $A(G)$ is hyperarchimedean, $A(G)\beta \subseteq A(G\beta)$ is always valid. Conversely, if G is representable and $A(G)$ is not hyperarchimedean, let P be a minimal but not maximal prime l -ideal of $A(G)$. Then $A(G)/P$ is not archimedean. Now, let Q be the minimal prime l -subgroup of G such that $Q \cap A(G) = P$. Since G is represent-

able, Q is normal in G and $(A(G)+Q)/Q \simeq A(G)/P$ is not archimedean. Thus, if G is representable, $A(G)\beta \subseteq A(G\beta)$ for all l -homomorphisms β if and only if $A(G)$ is hyperarchimedean. The following theorem gives an indication of what may happen if G is not representable.

Theorem 2.3. (1) *Let G be an l -group such that for every l -homomorphism β , $A(G)\beta \subseteq A(G\beta)$. Then for each characteristic convex l -subgroup K of $A(G)$, $A(G)/K$ is archimedean.*

(2) *Conversely, if A is an archimedean l -group such that for each characteristic convex l -subgroup K of A , A/K is archimedean, then there is an l -group G such that $A(G) \simeq A$ for every l -homomorphism β , $A(G)\beta \subseteq A(G\beta)$.*

PROOF. (1) Since $A(G)$ is characteristic in G , and K is characteristic in $A(G)$, K is normal (in fact, characteristic) in G . Thus $A(G)/K \subseteq A(G/K)$ so $A(G)/K$ is archimedean.

(2) Conversely, suppose A is an archimedean l -group such that A/K is archimedean for all characteristic convex l -subgroups K of A . Let $\mathfrak{A}$ be the l -automorphism group of A and let F be a free group with epimorphism $\mu: F \rightarrow \mathfrak{A}$. Since F is free, it is orderable (see [9], page 49) so we will view F as an ordered group. Let G be the splitting extension of A by F via μ , lexicographically ordered. I.e., the underlying set is $F \times A$ with group operation $(f, a) + (g, b) = (f + g, a(g\mu) + b)$ ordered by $(f, a) \geq 0$ if $f > 0$ or $f = 0$ and $a \geq 0$. It is easy to show that G is an l -group. $A(G) \simeq A$, and if P is an l -ideal of G , $P \cap A(G)$ is a characteristic convex l -subgroup of $A(G)$ so $A(G)\beta \subseteq A(G\beta)$ for all l -homomorphisms β .

Since the archimedean kernel does not have many of the nice properties of the hyperarchimedean kernel, (see [12]), we also considered the possibility of a maximal subdirect product of subgroups of the totally ordered group of real numbers. Unfortunately, as example 2 in section 4 shows, this does not, in general exist.

3. *The Archimedean kernel sequence.* Following the construction of the ascending central series for groups, we have

$$A(G) = A^1(G) \subseteq A^2(G) \subseteq A^3(G) \subseteq \dots$$

where, for an ordinal β ,

$$A^{\beta+1}(G)/A^\beta(G) = A(G/A^\beta(G)),$$

and if β is a limit ordinal,

$$A^\beta(G) = \bigcup_{\mu < \beta} A^\mu(G).$$

Let $a, b \in G^+$. We say a is *archimedean less than* b , denoted $a \ll b$, if $na \leq b$ for all integers $n > 0$. Thus, $0 < a \in A(G)$ if and only if $a \gg b$ implies $b = 0$. It seems reasonable to suppose $0 < a \in A^2(G)$ if and only if $a \gg b \gg c$ implies $c = 0$. In fact, it is easy to show that if $0 < a \in A^2(G)$ then $a \gg b \gg c$ does imply $c = 0$, but the converse is false, as example 3 shows. We do not know if there is an element-wise characterization of $A^2(G)$.

By a simple cardinality argument, there is an ordinal β (depending on G) so that $A^\beta(G) = A^{\beta+1}(G)$. Let $A^*(G) = A^\beta(G)$. We now investigate conditions for $A^*(G) = G$.

Theorem 3.1. $A^*(G) \subseteq [A(G)]''$. Thus if $A^*(G) = G$, then $A(G)$ is dense in G .

PROOF. It suffices to show $A^\gamma(G) \cap [A(G)]' = 0$ for each ordinal γ . We will induct on γ . The case when $\gamma = 1$ or γ a limit ordinal is clear. Suppose $A^{\gamma-1}(G) \cap [A(G)]' = 0$ and let $0 \leq x \in A^\gamma(G) \cap [A(G)]'$. Suppose (by way of contradiction) $x \neq 0$. Then $x \notin A^{\gamma-1}(G) \subseteq A(G)$, so there is a $0 < y \ll x$. Thus $x + A^{\gamma-1}(G) \gg y + A^{\gamma-1}(G)$ so $y \in A^{\gamma-1}(G)$. Since $0 < y < x$, $y \in [A(G)]'$ so $y \in [A(G)]' \cap A^{\gamma-1}(G) = 0$, a contradiction. Thus $A^\gamma(G) \cap [A(G)]' = 0$.

This condition is not sufficient since if $H = V(\Delta, R)$ (see [7]) where Δ is totally ordered and R is the totally ordered group of real numbers, then $A(H) = 0$ if and only if Δ has no minimal element. Let B be an archimedean l -group. The lex-extension G of B by H is an l -group with $A(G) \simeq B$ dense in G but if Δ has no minimal element, $A^2(G) = A(G) \neq G$.

Lemma 3.2. If K is a convex l -subgroup of G , then $K \cap A^*(G) = A^*(K)$.

The proof of this is similar to the proof of Proposition 1.4 of [10] and is omitted.

Theorem 3.3. Let G be a normal-valued l -group and suppose $\Gamma(G)$ satisfies the descending chain condition. Then $A^*(G) = G$.

PROOF. We will show that the descending chain condition on $\Gamma(G)$ forces each sequence of the form

$$x_1 \gg x_2 \gg \dots, x_i > 0$$

to be finite. This will guarantee that $A(G) \neq 0$ and a simple cardinality argument will show $A^\beta(G) = G$. Suppose (by way of contradiction) that there is an infinite sequence

$$x_1 \gg x_2 \gg x_3 \gg \dots, x_i > 0. \quad (\omega)$$

Since G is normal-valued, each value of x_{i+1} is properly contained in a value of x_i . The set $\{x_i | 1 \leq i\}$ is a filter in G^+ and hence is contained in an ultrafilter $\mathcal{J}$. $G^+ \setminus \mathcal{J}$ is the positive cone of a prime l -subgroup P . Since the collection of all convex l -subgroups which contain P is totally ordered, and $x_i \notin P$ for all i , there is a chain

$$G_1 \supset G_2 \supset G_3 \supset \dots,$$

where G_i is a value of x_i and each containment is proper. This contradicts the fact that $\Gamma(G)$ satisfies the descending chain condition. Therefore any sequence of the form (ω) is finite.

This condition is not necessary since an infinite cardinal product of the l -group of real numbers is an archimedean l -group and has minimal primes which are not values and therefore does not satisfy the descending chain condition, so $A(G) = G$ but $\Gamma(G)$ does not satisfy the descending chain condition.

4. Examples. 1. An example of an abelian l -group G with a basis such that $A(G)^X \neq A(G^X)$ for $X = P$ or SP .

Let G be the set of all real-valued sequences with domain the non-negative integers such that $f \in G$ implies there exists an n so that if $m \geq n$, then $f(0) = f(m)$. When G has pointwise addition and partial order $f \geq g$ if $f(0) > g(0)$ and $f(n) \geq g(n)$ for $n > 1$ or $f(0) = g(0)$ and $f(n) \geq g(n)$ for $n \geq 1$, then G is an l -group. Then

$A(G) = \{f \in G \mid f(0) = 0\}$ which is an SP -group. G^{SP} is the l -group of bounded functions and G^P is the l -group of eventually constant sequences. $A(G^{SP}) = \{f \in G^{SP} \mid f(0) = 0\}$ and $A(G^P) = \{f \in G^P \mid f(0) = 0\}$.

2. An example of an archimedean l -group G which has no convex l -subgroup which is maximal with respect to being a subdirect product of subgroups of the l -group of real numbers. This example is due to SHELDON [14].

Let F be the l -subgroup of $\prod_{[0,1]} \mathbb{Z}$ generated by the characteristic functions of closed intervals, where $\mathbb{Z}$ is the integers. For each $b \in [0, 1]$, let

$$h_b(x) = \begin{cases} [1/(x-b)^2] & x \neq b \\ 0 & x = b \end{cases}$$

where $[...]$ is the greatest integer function. Let K be the subgroup of $\prod \mathbb{Z}$ generated by the h_b , $b \in [0, 1]$ and F . Then any $f \in K$ can be written in the form

$$f = k + n_1 h_{b_1} + n_2 h_{b_2} + \dots + n_m h_{b_m} \quad (\zeta)$$

where $k \in F$ and each $n_j \neq 0$. Let $G = K/\Sigma \mathbb{Z}$. Then

1. G is an l -subgroup of $\prod \mathbb{Z}/\Sigma \mathbb{Z}$.
2. G is archimedean.
3. The prime l -ideals of G are

$$P_b = \{f + \Sigma \mathbb{Z} \in G \mid h_b \text{ does not occur in the expansion } (\zeta) \text{ for } f\}$$

$$P_b^l = \{f + \Sigma \mathbb{Z} \in G \mid f(x) = 0 \text{ for almost all } x \in (y, b) \text{ some } y < b\}, (b \neq 0)$$

$$P_b^r = \{f + \Sigma \mathbb{Z} \in G \mid f(x) = 0 \text{ for almost all } x \in (b, z) \text{ some } z > b\}, (b \neq 1)$$

4. $P_b \supseteq P_b^l \cup P_b^r$.
 5. $F/\Sigma \mathbb{Z} \subseteq \bigcap P_b$, so G is not a subdirect sum of subgroups of the real numbers.
 6. If H is a convex l -subgroup of G , then H is a subdirect sum of subgroups of the reals if and only if
 $S = \{b \in [0, 1] \mid P_b \cap H \subset H \text{ has empty interior}\}.$
- 1—5 are in [14].

Proof of 6. Since H is convex, $\Gamma(H)$ is precisely $\{K \in \Gamma(G) \mid K \cap H \subset H\}$.

($\rightarrow$) Suppose S has non-empty interior, say $(s, t) \subseteq S$. Let $s < m < n < t$. Since $[m, n] \subseteq (s, t) \subseteq S$, we can find a $g \in H$ so that $g(x) \equiv 1$ for all $x \in [m, n]$, and $g(x) = 0$ if $x \notin (s, t)$. Now this g is in every maximal prime of H so H is not a subdirect sum of real numbers.

($\leftarrow$) Suppose (by way of contradiction) S has empty interior. If $\Sigma \mathbb{Z} < \Sigma \mathbb{Z} + g \in H$, then there is an interval $(s, t) \subset [0, 1]$ such that $g(x) \equiv 1$ for almost all $x \in (s, t)$. Since S has empty interior, there is a $b \in (s, t)$ such that $b \notin S$, so $P_b \cap H = H$. Since $g(x) \equiv 1$ for almost all $x \in (s, b)$, $g + \Sigma \mathbb{Z} \notin P_b^r$ which is a maximal prime of H . Thus H is a subdirect product of subgroups of R .

Since $[0, 1]$ has no subsets maximal with respect to having empty interior, G has no maximal subdirect product of subgroups of R .

3. An example of an l -group such that for all $a \in G$, $a \gg b \gg c$ implies $c = 0$, but $A^2(G) \neq G$.

Let G be the set of all real sequences $(a_0, a_1, a_2, \dots)$ such that $a_n = na_0 + a_1$ for

all but finitely many $n \geq 2$ with pointwise addition and ordered by $(a_0, a_1, a_2, \dots) \geq 0$ if $a_n \geq 0$ for $n > 2$ and $a_0 > 0$, or $a_0 = 0$ and $a_1 > 0$, or $a_0 = a_1 = 0$ and $a_2 \geq 0$. It is easy to see that $A(G)$ is the set of all sequences with $a_0 = a_1 = 0$ and $G/A(G) = \overrightarrow{R \oplus R}$.

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