

On birecurrent Kähler manifold

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In this paper we have studied the properties of Bochner curvature tensor in a Kähler manifold by making use of birecurrent condition.

1. Introduction.

Agreement 1.1. In what follows the equations will hold for arbitrary C^∞ vector fields $X, Y, Z, T, W, \dots$, etc. whenever they occur.

We consider a $2n$ -dimensional real manifold M_{2n} of differentiability class C^{r+1} . Let F be the vector valued linear function defined in M_{2n} , such that

$$(1.1) \quad \bar{X} + X = 0,$$

for arbitrary vector field X in M_{2n} , where

$$(1.2) \quad \bar{X} \stackrel{\text{def}}{=} F(X).$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold.

If the Hermite metric g :

$$(1.3) \quad g(\bar{X}, \bar{Y}) = g(X, Y),$$

be defined in almost complex manifold, then M_{2n} is called an almost Hermite manifold.

Let us put

$$(1.4) \quad 'F(X, Y) = g(\bar{X}, Y).$$

Then from (1.1), (1.2), (1.3), and (1.4), we have

$$(1.5)a \quad 'F(\bar{X}, \bar{Y}) = 'F(X, Y),$$

$$(1.5)b \quad 'F(X, Y) + 'F(Y, X) = 0.$$

Suppose D is a Riemannian connexion satisfying:

$$(1.6)a \quad D_X Y - D_Y X = [X, Y],$$

$$(1.6)b \quad (D_X g)(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z).$$

If in addition to (1.2), (1.3) and (1.6) the condition

$$(1.7) \quad (D_X F)(Y) = 0$$

is satisfied, then M_{2n} is called Kähler manifold.

Let K be the curvature tensor of M_{2n} given by

$$(1.8) \quad K(X, Y, Z) \stackrel{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z.$$

Let Ric be the Ricci tensor of M_{2n} given by

$$(1.9) \quad \text{Ric}(Y, Z) \stackrel{\text{def}}{=} (C'_1 K)(Y, Z),$$

and

$$(1.10) \quad \text{Ric}(Y, Z) \stackrel{\text{def}}{=} \text{Ric}(Z, Y).$$

Let us put

$$(1.11) \quad \text{Ric}(Y, Z) \stackrel{\text{def}}{=} g(r(Y), Z)$$

and

$$(1.12) \quad R \stackrel{\text{def}}{=} (C'_1 r),$$

where R is the scalar curvature.

We know that a manifold M_{2n} is called birecurrent manifold (TAKENO, 1971) if

$$(1.13) \quad (\nabla \nabla K)(Z, T, W, X, Y) = a(X, Y) K(Z, T, W),$$

where $a(X, Y)$ is a non-vanishing C^∞ function.

The manifold M_{2n} is called Ricci birecurrent if

$$(1.14)a \quad (\nabla \nabla \text{Ric})(Z, T, X, Y) = a(X, Y) \text{Ric}(Z, T),$$

which implies

$$(1.14)b \quad (\nabla \nabla r)(Z, X, Y) = a(X, Y) r(Z),$$

and

$$(1.14)c \quad (\nabla \nabla R)(X, Y) = a(X, Y) R,$$

where

$$(1.14)d \quad (D_X K)(Z, T, W) = (\nabla K)(Z, T, W, X).$$

The holomorphic sectional curvature K of M_{2n} with regard to X is given by (MISHRA, [1])

$$(1.15) \quad Kg(X, X)g(X, X) + K(X, \bar{X}, X, \bar{X}) = 0.$$

The projective curvature tensor W^* , the conformal curvature tensor V , the conharmonic curvature tensor L , the concircular curvature tensor C , H -projective curvature tensor P , T -concircular curvature tensor T^* are given by

$$(1.16) \quad W^*(Z, T, W) = K(Z, T, W) - \frac{1}{(2n-1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T],$$

$$(1.17) \quad V(Z, T, W) =$$

$$= K(Z, T, W) - \frac{1}{(2n-1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + r(Z)g(T, W) - r(T)g(Z, W)] + \\ + R/2(2n-1)(n-1)[g(T, W)Z - g(Z, W)T],$$

$$(1.18) \quad L(Z, T, W) =$$

$$= K(Z, T, W) - \frac{1}{2(n-1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + r(Z)g(T, W) - r(T)g(Z, W)],$$

$$(1.19) \quad C(Z, T, W) = K(Z, T, W) - \frac{R}{2n(2n-1)} [g(T, W)Z - g(Z, W)T],$$

$$(1.20) \quad P(Z, T, W) = K(Z, T, W) - \\ - \frac{1}{2(n+1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + \text{Ric}(\bar{T}, W)\bar{Z} + \text{Ric}(Z, \bar{W})\bar{T} + 2\text{Ric}(Z, \bar{T})\bar{W}],$$

$$(1.21) \quad T^*(Z, T, W) = \\ = K(Z, T, W) + \frac{R}{4n(n+1)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + g(\bar{Z}, T)\bar{W}],$$

respectively.

Definition 1.1. Let Q be a vector valued trilinear function, by any one of the curvature tensor W^*, V, L, C, P, T^* . Then the Kähler manifold M_{2n} is said to be Q -birecurrent (RATHORE and MISHRA, 1973) if

$$(1.22) \quad (\nabla \nabla Q)(Z, T, W, X, Y) = a(X, Y)Q(Z, T, W).$$

2. The Bochner Curvature tensor.

The Bochner curvature tensor B in Kähler manifold M_{2n} is given by

$$(2.1) \quad B(Z, T, W) = K(Z, T, W) + \frac{1}{2(n+2)} [\text{Ric}(Z, W) - \text{Ric}(T, W)Z + g(Z, W)r(T) - \\ - g(T, W)r(Z) + \text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + \\ + 2\text{Ric}(\bar{Z}, T)\bar{W} + 2g(\bar{Z}, T)r(\bar{W})] - \\ - \frac{R}{4(n+1)(n+2)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}].$$

Let us put

$$(2.2) \quad {}^*B(Z, T, W, U) \stackrel{\text{def}}{=} g(B(Z, T, W), U).$$

Then we have

(2.3)

$$\begin{aligned} {}'B(Z, T, W, U) = {}'K(Z, T, W, U) + \frac{1}{2(n+2)} [g(T, U) \text{Ric}(Z, W) - g(Z, U) \text{Ric}(T, W) + \\ + g(Z, W) \text{Ric}(T, U) - g(T, W) \text{Ric}(Z, U) + \text{Ric}(\bar{Z}, W)g(\bar{T}, U) - \text{Ric}(\bar{T}, W)g(\bar{Z}, U) + \\ + g(\bar{Z}, W) \text{Ric}(\bar{T}, U) - g(\bar{T}, W) \text{Ric}(\bar{Z}, U) + \\ + 2 \text{Ric}(\bar{Z}, T)g(\bar{W}, U) + 2g(\bar{Z}, T) \text{Ric}(\bar{W}, U)] - \\ - \frac{R}{4(n+1)(n+2)} [g(Z, W)g(T, U) - g(T, W)g(Z, U) + g(\bar{Z}, W)g(\bar{T}, U) - \\ - g(\bar{T}, W)g(\bar{Z}, U) + 2g(\bar{Z}, T)g(\bar{W}, U)]. \end{aligned}$$

Definition 2.1. The Kähler manifold M_{2n} will be called a Bochner birecurrent manifold if

$$(2.4) \quad (\nabla \nabla B)(Z, T, W, X, Y) + a(X, Y)B(Z, T, W).$$

Now we have following theorems:

Theorem 2.1. *If a Kähler manifold is a Bochner birecurrent manifold and a Ricci birecurrent manifold for the same recurrence parameter, it is also a birecurrent manifold.*

PROOF. From (2.1), we have

(2.5)

$$\begin{aligned} (\nabla \nabla B)(Z, T, W, X, Y) = (\nabla \nabla K)(Z, T, W, X, Y) + \frac{1}{2(n+2)} [(\nabla \nabla \text{Ric})(Z, W, X, Y)T - \\ - (\nabla \nabla \text{Ric})(T, W, X, Y)Z + (\nabla \nabla r)(T, X, Y)g(Z, W) - (\nabla \nabla r)(Z, X, Y)g(T, W) + \\ + (\nabla \nabla \text{Ric}(\bar{Z}, W, X, Y)\bar{T} - (\nabla \nabla \text{Ric})(\bar{T}, W, X, Y)\bar{Z} + (\nabla \nabla r)(\bar{T}, X, Y)g(\bar{Z}, W) - \\ - (\nabla \nabla r)(\bar{Z}, X, Y)g(\bar{T}, W) + 2(\text{Ric})(\bar{Z}, T, X, Y)\bar{W} + 2(\nabla \nabla r)(\bar{W}, X, Y)g(\bar{Z}, T)] - \\ - \frac{(\nabla \nabla R)(X, Y)}{4(n+1)(n+2)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}]. \end{aligned}$$

Let the Kähler manifold M_{2n} be Ricci birecurrent and Bochner birecurrent manifold. Then by using (1.14)a, (1.14)b, (1.14)c and (2.4) in (2.5) we get

(2.6)

$$\begin{aligned} (\nabla \nabla K)(Z, T, W, X, Y) = \\ = a(X, Y) [B(Z, T, W) - \frac{1}{2(n+2)} \{ \text{Ric}(Z, W)T - \text{Ric}(T, W)Z + r(T)g(Z, W) - \\ - r(Z)g(T, W) + \text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + r(\bar{T})g(\bar{Z}, W) - r(\bar{Z})g(\bar{T}, W) + \\ + 2 \text{Ric}(\bar{Z}, T)\bar{W} + 2r(\bar{W})g(\bar{Z}, T) \} - \\ - \frac{R}{4(n+1)(n+2)} \{ g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W} \}]. \end{aligned}$$

Using equation (2.1) in the above equation, we have

$$(2.7) \quad (\nabla \nabla K)(Z, T, W, X, Y) = a(X, Y)K(Z, T, W),$$

that is, Kähler manifold M_{2n} is a birecurrent manifold, which proves the statement.

Similarly, we can prove that if a Kähler manifold is birecurrent and either Bochner birecurrent or Ricci birecurrent manifold for the same recurrence parameter, i.e. also either Ricci birecurrent or Bochner birecurrent manifold respectively.

Theorem 2.2. *We have*

$$\begin{aligned} B(Z, T, W) &= W^*(Z, T, W) + \frac{5}{2(2n-1)(n+2)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T] + \\ &+ \frac{1}{2(n+2)} [g(Z, W)r(T) - g(T, W)r(Z) + \text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + \\ &+ g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + 2\text{Ric}(\bar{Z}, T)\bar{W} + 2g(\bar{Z}, T)r(\bar{W})] - \\ &- \frac{R}{4(n+1)(n+2)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}]. \end{aligned}$$

Hence, if a Kähler manifold is a Ricci birecurrent and a Bochner birecurrent manifold for the same recurrence parameter, it is also a projective birecurrent manifold.

Theorem 2.3. *We have*

$$\begin{aligned} (2.9) \quad B(Z, T, W) &= V(Z, T, W) + \frac{5}{2(n+2)(2n-1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + \\ &+ r(Z)g(T, W) - r(T)g(Z, W)] + \frac{1}{2(n+2)} [\text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + \\ &+ g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + 2\text{Ric}(\bar{Z}, T)\bar{W} + 2g(\bar{Z}, T)r(\bar{W}) - \\ &- \frac{3R(3n-1)}{4(2n-1)(n^2-1)(n+2)} [g(T, W)Z - g(Z, W)T] - \\ &- \frac{R}{4(n+1)(n+2)} [g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}]. \end{aligned}$$

Hence, if a Kähler manifold is a Ricci birecurrent and a Bochner birecurrent manifold for the same recurrence parameter, it is also a conformal birecurrent manifold.

Theorem 2.4. *We have*

$$\begin{aligned} (2.10) \quad B(Z, T, W) &= L(Z, T, W) + \frac{3}{2(n+2)(n-1)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + \\ &+ r(Z)g(T, W) - r(T)g(Z, W)] + \frac{1}{2(n+2)} [\text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + \\ &+ g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + 2\text{Ric}(\bar{Z}, T)\bar{W} + 2g(\bar{Z}, T)r(\bar{W})] - \\ &- \frac{R}{4(n+1)(n+2)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}]. \end{aligned}$$

Hence, if a Kähler manifold is a Ricci bi-recurrent and a Bochner bi-recurrent manifold for the same recurrence parameter it is also a conharmonic bi-recurrent manifold.

Theorem 2.5. *We have*

$$(2.11) \quad B(Z, T, W) = C(Z, T, W) + \frac{1}{2(n+2)} [\text{Ric}(Z, W)T - \text{Ric}(T, W)Z + \\ + g(Z, W)r(T) - g(T, W)r(Z) + \text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + \\ + g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + 2\text{Ric}(\bar{Z}, T) + 2g(\bar{Z}, T)r(\bar{W})] + \\ + \frac{(4n^2 + 5n + 4)R}{4n(2n-1)(n+1)(n+2)} [g(T, W)Z - g(Z, W)T] - \\ - \frac{R}{4(n+1)(n+2)} [g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}].$$

Hence, if a Kähler manifold is a Ricci bi-recurrent manifold and a Bochner bi-recurrent manifold for the same recurrence parameter it is also a concircular bi-recurrent manifold.

Theorem 2.6. *We have*

$$(2.12) \quad B(Z, T, W) = P(Z, T, W) + \frac{1}{2(n+1)(n+2)} [\text{Ric}(T, W)Z - \text{Ric}(Z, W)T + \\ + \text{Ric}(\bar{T}, W)\bar{Z} + \text{Ric}(Z, \bar{W})\bar{T} + 2\text{Ric}(Z, \bar{T})\bar{W}] + \frac{1}{2(n+2)} [g(Z, W)r(T) - \\ - g(T, W)r(Z) + g(\bar{Z}, W)r(\bar{T}) - g(\bar{T}, W)r(\bar{Z}) + 2g(\bar{Z}, T)r(\bar{W})] - \\ - \frac{R}{4(n+1)(n+2)} [g(Z, W)T - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + 2g(\bar{Z}, T)\bar{W}].$$

Hence, if a Kähler manifold is a Ricci bi-recurrent manifold and a Bochner bi-recurrent manifold for the same recurrence parameter, it is also a H-projective bi-recurrent manifold.

Theorem 2.7. *We have*

$$(2.13) \quad B(Z, T, W) = T^*(Z, T, W) + \frac{1}{2(n+2)} [\text{Ric}(Z, W)T - \text{Ric}(T, W)Z + \\ + g(Z, W)r(T) - g(T, W)r(Z) + \text{Ric}(\bar{Z}, W)\bar{T} - \text{Ric}(\bar{T}, W)\bar{Z} + g(\bar{Z}, W)r(\bar{T}) - \\ - g(\bar{T}, W)r(\bar{Z}) + 2\text{Ric}(\bar{Z}, T)\bar{W} + 2g(\bar{Z}, T)r(\bar{W})] - \frac{R}{2n(n+2)} [g(Z, W)T - \\ - g(T, W)Z + g(\bar{Z}, W)\bar{T} - g(\bar{T}, W)\bar{Z} + g(\bar{Z}, T)\bar{W}] - \frac{R}{4(n+1)(n+2)} [g(\bar{Z}, T)\bar{W}].$$

Hence, if a Kähler manifold is a Ricci birecurrent manifold and a Bochner birecurrent manifold for the same recurrence parameter, it is also a T -concircular birecurrent manifold.

3. Holomorphic sectional curvature:

Theorem (3.1). The holomorphic sectional curvature k of a Kähler manifold M_{2n} with respect to a vector X is given by, in terms of Bochner curvature tensor,

$$(3.1) \quad k = -\frac{{}'B(X, \bar{X}, X, \bar{X})}{g(X, X)g(X, X)} + \frac{4 \operatorname{Ric}(X, X)}{(n+2)g(X, X)} - \frac{R}{(n+1)(n+2)}.$$

PROOF. By putting $Z=X$, $T=\bar{X}$, $W=X$ and $U=\bar{X}$ in (2.3) and using (1.1), (1.3) and (1.4), we get

$$(3.2) \quad {}'B(X, \bar{X}, X, \bar{X}) = {}'K(X, \bar{X}, X, \bar{X}) + \frac{4}{(n+2)} g(X, X) \operatorname{Ric}(X, X) - \frac{Rg(X, X)g(X, X)}{(n+1)(n+2)}$$

Substituting $'K(X, \bar{X}, X, \bar{X})$ from (1.15) in (3.2), we get

$$(3.3) \quad \begin{aligned} {}'B(X, \bar{X}, X, \bar{X}) &= \\ &= -kg(X, X)g(X, X) + \frac{4}{(n+2)} g(X, X) \operatorname{Ric}(X, X) - \frac{Rg(X, X)g(X, X)}{(n+1)(n+2)} \end{aligned}$$

which proves the statement.

Corollary 3.1. The holomorphic sectional curvature k in the direction of the unit vector X is given by

$$(3.4) \quad k = -{}'B(X, \bar{X}, X, \bar{X}) + \frac{4}{n+2} \operatorname{Ric}(X, X) - \frac{R}{(n+1)(n+2)}.$$

PROOF. We have $g(X, X)=1$.

Using it in (3.1), we get (3.4).

Theorem 3.2. For a Kähler manifold of constant curvature, we have

$$(3.5) \quad \begin{aligned} {}'B(Z, T, W, U) &= \frac{3k}{2(n+1)} [g(Z, U)g(T, W) - g(Z, W)g(T, U)] + \\ &+ \frac{(2n-1)k}{(n+1)} [g(\bar{Z}, W)g(\bar{T}, U) - g(\bar{Z}, U)g(\bar{T}, W) + 2g(\bar{Z}, T)g(\bar{W}, U)]. \end{aligned}$$

PROOF. If the Kähler manifold is of constant curvature k , then we know that

$$(3.6) \quad {}'K(Z, T, W, U) = k[g(Z, U)g(T, W) - g(Z, W)g(T, U)],$$

$$(3.7) \quad K(Z, T, W) = k[g(T, W)Z - g(Z, W)T],$$

$$(3.8) \quad \operatorname{Ric}(T, W) = k[2ng(T, W) - g(T, W)],$$

and

$$(3.9) \quad r(T) = (2n-1)kT.$$

Contracting (3.9), we have

$$(3.10) \quad R = 2n(2n-1)k.$$

Using (3.6), (3.8) and (3.10), we get (3.5).

Corollary (3.2). If a Kähler manifold of constant curvature is flat, then Bochner curvature tensor vanishes.

PROOF. By putting $k=0$ in (3.5), we get

$${}^*B(Z, T, W, U) = 0.$$

Hence the statement.

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