

Projective symmetry in a special Kawaguchi space

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1. Summary and introduction

M. C. CHAKI and B. GUPTA [1] have defined an n -dimensional conformally symmetric Riemannian space. In one of her papers, B. GUPTA [2] has studied various properties of the projectively symmetric Riemannian spaces. The object of this paper is to define an n -dimensional projectively symmetric special Kawaguchi space and discuss its properties.

An n -dimensional metric space, in which the arc length of a curve $x^i = x^i(t)$ is given by the special integral

$$(1.1) \quad s = \int F^{1/p} dt, \quad F \equiv A_i(x, x') x''^i + B(x, x'),$$

where $A_i(x, x')$ and $B(x, x')$ are differentiable functions of x, x' , is called an n -dimensional special Kawaguchi space. A. KAWAGUCHI ([3], [4]) has defined a connection in a special Kawaguchi space by applying the "Craig vector" of the function $A_i x''^i + B$. The Craig vector is defined as

$$(1.2) \quad T_i = (A_{k(i)} - 2A_{i(k)}) x''^k - 2A_{ik} x'^k + B_{(i)},$$

where
$$A_{k(i)} = \frac{\partial A_k}{\partial x'^i}, \quad A_{ik} = \frac{\partial A_i}{\partial x'^k}, \quad B_{(i)} = \frac{\partial B}{\partial x'^i}.$$

There exists a relation

$$(1.3) \quad \begin{cases} F = A_i x'^{[2]i}, & \text{where } 2p \neq 3, \\ x'^{[2]i} = x''^i + 2\Gamma^i, & 2\Gamma^i = (2A_{ik} x'^k - B_{(i)}) G^{ii}, \\ G_{ik} = 2A_{i(k)} - A_{k(i)}, & G_{ik} G^{ii} = \delta_k^i. \end{cases}$$

The covariant derivative of a contravariant vector field v^i , which is homogeneous of degree zero with respect to x'^i , is defined by

$$(1.4) \quad \nabla_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^k} \Gamma_{(j)}^k + \Gamma_{(k)(j)}^i v^k, \quad \nabla'_j v^i = \frac{\partial v^i}{\partial x'^j}.$$

Then from the brackets of Poisson for covariant derivatives, we find the curvature tensor fields as follows:

$$(1.5) \quad (\nabla_j \nabla_k - \nabla_k \nabla_j) v^i = -R_{jkl}^i v^l + K_{jk}^i \nabla_l' v^l,$$

$$(1.6) \quad (\nabla_j \nabla'_k - \nabla'_k \nabla_j) v^i = -B_{jkl}^i v^l,$$

where

$$(1.7) \quad B_{jkl}^{\cdot\cdot\cdot i} = \Gamma_{(j)(k)(l)}^i,$$

$$(1.8)$$

$$(1.9) \quad R_{jkl}^{\cdot\cdot\cdot i} = \frac{\partial \Gamma_{(l)(j)}^i}{\partial x^k} - \frac{\partial \Gamma_{(l)(k)}^i}{\partial x^j} + \Gamma_{(l)(j)}^h \Gamma_{(k)(h)}^i - \Gamma_{(l)(k)}^h \Gamma_{(j)(h)}^i + \Gamma_{(j)}^h \Gamma_{(l)(k)(h)}^i - \Gamma_{(k)}^h \Gamma_{(l)(j)(h)}^i,$$

$$K_{jk}^{\cdot\cdot i} = \frac{\partial \Gamma_{(j)}^i}{\partial x^k} - \frac{\partial \Gamma_{(k)}^i}{\partial x^j} + \Gamma_{(j)}^h \Gamma_{(k)(h)}^i - \Gamma_{(k)}^h \Gamma_{(j)(h)}^i.$$

These curvature tensor fields satisfy the following identities:

$$(1.10) \quad R_{jkl}^{\cdot\cdot\cdot i} = -R_{kjl}^{\cdot\cdot\cdot i}, \quad R_{jkl}^{\cdot\cdot\cdot i} + R_{kjl}^{\cdot\cdot\cdot i} + R_{ljk}^{\cdot\cdot\cdot i} = 0,$$

$$(1.11) \quad K_{jk}^{\cdot\cdot i} = R_{jkl}^{\cdot\cdot\cdot i} x'^l, \quad R_{jkl}^{\cdot\cdot\cdot i} = K_{jk(l)}^{\cdot\cdot i} = \nabla'_l K_{jk}^{\cdot\cdot i},$$

$$(1.12) \quad B_{jkl}^{\cdot\cdot\cdot i} x'^l = 0.$$

The Bianchi identity for $K_{jk}^{\cdot\cdot i}$ is given by

$$(1.13) \quad \nabla_h K_{jk}^{\cdot\cdot i} + \nabla_j K_{kh}^{\cdot\cdot i} + \nabla_k K_{hj}^{\cdot\cdot i} = 0.$$

2. Projective curvature tensor

S. KAWAGUCHI [5] has defined the projective curvature tensor W_{jkl}^i in a special Kawaguchi space as follows:

$$(2.1) \quad W_{jkl}^i = R_{jkl}^{\cdot\cdot\cdot i} - \frac{1}{n-1} [\delta_j^i R_{akl}^{\cdot\cdot\cdot a} - \delta_k^i R_{ajl}^{\cdot\cdot\cdot a}] - \frac{x'^i}{n+1} \nabla'_l R_{jka}^{\cdot\cdot\cdot a} +$$

$$+ \frac{x'^h}{n^2-1} [\delta_j^i \nabla'_l R_{hka}^{\cdot\cdot\cdot a} - \delta_k^i \nabla'_l R_{hja}^{\cdot\cdot\cdot a}] - \frac{1}{n^2-1} [\delta_j^i R_{kla}^{\cdot\cdot\cdot a} - \delta_k^i R_{jla}^{\cdot\cdot\cdot a}] - \frac{1}{n+1} \delta_l^i R_{jka}^{\cdot\cdot\cdot a}.$$

Thus defined projective curvature tensor is a homogeneous function of degree zero with respect to x'^i and satisfies the following relations:

$$(2.2) \quad W_{akl}^a = W_{kal}^a = W_{kla}^a = 0,$$

$$(2.3) \quad \nabla'_a W_{jkl}^a = 0,$$

$$(2.4) \quad W_{jkl}^i + W_{kjl}^i = 0, \quad W_{jkl}^i + W_{klj}^i + W_{ljk}^i = 0,$$

$$(2.5) \quad W_{jkl}^i x'^l = W_{jk}^i, \quad \nabla'_l W_{jk}^i = W_{jkl}^i,$$

where

$$(2.6) \quad W_{jk}^i = K_{jk}^{\cdot\cdot i} + \frac{x'^i}{n+1} [\nabla'_j K_{ka}^{\cdot\cdot\cdot a} - \nabla'_k K_{ja}^{\cdot\cdot\cdot a}] + \frac{\delta_j^i}{n+1} [K_{ka}^{\cdot\cdot\cdot a} + \frac{1}{n-1} \nabla'_k (K_{ab}^{\cdot\cdot\cdot a} x'^b)] -$$

$$\frac{\delta_k^i}{n+1} [K_{ja}^{\cdot\cdot\cdot a} + \frac{1}{n-1} \nabla'_j (K_{ab}^{\cdot\cdot\cdot a} x'^b)].$$

3. Projectively symmetric special Kawaguchi space

We call an n -dimensional special Kawaguchi space a projectively symmetric special Kawaguchi space if the projective curvature tensor W_{jkl}^i defined by (2.1) satisfies the relation

$$(3.1) \quad \nabla_m W_{jkl}^i = 0.$$

For brevity, we denote such a space by PSK_n -space. Applying ∇_m to (2.1) and using the commutation formula (1.6) and the fact that $\nabla_m x'^i = 0$, we find that the condition for a PSK_n -space becomes

$$(3.2) \quad \begin{aligned} & \nabla_m R_{jki}^{\cdot\cdot\cdot i} - \frac{1}{n-1} [\delta_j^i \nabla_m R_{aki}^{\cdot\cdot\cdot a} - \delta_k^i \nabla_m R_{aji}^{\cdot\cdot\cdot a}] - \\ & - \frac{x'^i}{n+1} [\nabla_l' \nabla_m R_{jka}^{\cdot\cdot\cdot a} + B_{mij}^{\cdot\cdot\cdot r} R_{rka}^{\cdot\cdot\cdot a} + B_{mik}^{\cdot\cdot\cdot r} R_{jra}^{\cdot\cdot\cdot a}] + \\ & + \frac{x'^h}{n^2-1} [\delta_j^i \{ \nabla_l' \nabla_m R_{hka}^{\cdot\cdot\cdot a} + B_{mih}^{\cdot\cdot\cdot r} R_{rka}^{\cdot\cdot\cdot a} \} - \delta_k^i \{ \nabla_l' \nabla_m R_{hja}^{\cdot\cdot\cdot a} + B_{mij}^{\cdot\cdot\cdot r} R_{hra}^{\cdot\cdot\cdot a} \}] - \\ & - \frac{1}{n^2-1} [\delta_j^i \nabla_m R_{kila}^{\cdot\cdot\cdot a} - \delta_k^i \nabla_m R_{jila}^{\cdot\cdot\cdot a}] - \frac{\delta_l^i}{n+1} \nabla_m R_{jka}^{\cdot\cdot\cdot a} = 0. \end{aligned}$$

If the space is recurrent, we have

$$(3.3) \quad \nabla_m R_{jki}^{\cdot\cdot\cdot i} = V_m R_{jki}^{\cdot\cdot\cdot i},$$

where V_m is a non-zero vector field. If $B_{jkl}^{\cdot\cdot\cdot i} = 0$, then V_m is independent of x'^i . Letting $B_{jki}^{\cdot\cdot\cdot i} = 0$ and using (3.3) in (3.2), we have

$$(3.4) \quad V_m W_{jkl}^i = 0.$$

Since V_m is a non-zero vector field, we have

$$(3.5) \quad W_{jkl}^i = 0,$$

i.e. the space is projectively flat. Thus we have

Theorem 3.1. *If $B_{jkl}^{\cdot\cdot\cdot i} = 0$, then every recurrent PSK_n -space is projectively flat.*
Now contracting (3.2) with respect to the indices i and m , we have

$$(3.6) \quad \begin{aligned} & \nabla_l R_{jki}^{\cdot\cdot\cdot i} - \frac{1}{n-1} [\nabla_j R_{aki}^{\cdot\cdot\cdot a} - \nabla_k R_{aji}^{\cdot\cdot\cdot a}] - \frac{x'^i}{n+1} \nabla_l' \nabla_i R_{jka}^{\cdot\cdot\cdot a} + \\ & + \frac{x'^h}{n^2-1} [\nabla_l' \nabla_j R_{hka}^{\cdot\cdot\cdot a} - \nabla_l' \nabla_k R_{hja}^{\cdot\cdot\cdot a}] - \frac{1}{n^2-1} [\nabla_j R_{kila}^{\cdot\cdot\cdot a} - \nabla_k R_{jila}^{\cdot\cdot\cdot a}] - \frac{1}{n+1} \nabla_l R_{jka}^{\cdot\cdot\cdot a} = 0. \end{aligned}$$

Transvecting (3.6) with x'^l , we have

$$(3.7) \quad \nabla_i K_{jk}^{\cdot\cdot\cdot i} - \frac{1}{n-1} [\nabla_j K_{ak}^{\cdot\cdot\cdot a} - \nabla_k K_{aj}^{\cdot\cdot\cdot a}] - \frac{x'^l}{n^2-1} [\nabla_j R_{kila}^{\cdot\cdot\cdot a} - \nabla_k R_{jila}^{\cdot\cdot\cdot a}] - \frac{x'^l}{n+1} \nabla_l R_{jka}^{\cdot\cdot\cdot a} = 0,$$

where we have used (1.11) and the fact that $R_{jkl}^{i\cdots i}$ is homogeneous of degree zero with respect to x'^i . Hence we have

Theorem 3.2. *In a PSK_n -space the relation (3.7) holds good.*

Let us now consider the projective transformation of the function $\Gamma^i(x, x')$ defined by the equation

$$(3.8) \quad \bar{\Gamma}^i(x, x') = \Gamma^i(x, x') + \alpha(x, x')x'^i,$$

where $\alpha(x, x')$ is an arbitrary scalar function positively homogeneous of the first degree with respect to x'^i . The tensor W_{jkl}^i is invariant under the projective change (3.8), i.e.,

$$(3.9) \quad \bar{W}_{jkl}^i = W_{jkl}^i.$$

Differentiating (3.9) covariantly and noting the homogeneity property of W_{jkl}^i , we have

$$(3.10) \quad \begin{aligned} \bar{\nabla}_m \bar{W}_{jkl}^i &= \nabla_m W_{jkl}^i - \alpha \nabla_m' W_{jkl}^i + W_{jkl}^a \alpha_{(a)} \delta_m^i + \\ &+ \alpha_{(a)(m)} W_{jkl}^a x'^i - \alpha_{(j)} W_{mkl}^i - W_{akl}^i \alpha_{(j)(m)} x'^a - W_{jml}^i \alpha_{(k)} - 2\alpha_{(m)} W_{jkl}^i - W_{jal}^i \alpha_{(k)(m)} x'^a - \\ &- W_{jkm}^i \alpha_{(l)} - W_{jka}^i \alpha_{(l)(m)} x'^a, \end{aligned}$$

where $\bar{\nabla}_m$ denotes covariant differentiation with respect to $\bar{\Gamma}^i$. Contracting (3.10) with respect to the indices i and m and using (2.2), (2.3) and the homogeneity property of $\alpha(x, x')$, we have

$$(3.11) \quad \begin{aligned} \bar{\nabla}_i \bar{W}_{jkl}^i &= \nabla_i W_{jkl}^i + (n-2)\alpha_{(a)} W_{jkl}^a - \\ &- x'^a [W_{akl}^i \alpha_{(j)(i)} + W_{jal}^i \alpha_{(k)(i)} + W_{jka}^i \alpha_{(l)(i)}]. \end{aligned}$$

We now assume that the space and its projective map are both projectively symmetric. Then we have

$$(3.12) \quad \nabla_m W_{jkl}^i = 0 \quad \text{and} \quad \bar{\nabla}_m \bar{W}_{jkl}^i = 0.$$

Consequently, from (3.11), we have

$$(3.13) \quad (n-2)\alpha_{(a)} W_{jkl}^a - x'^a [W_{akl}^i \alpha_{(j)(i)} + W_{jal}^i \alpha_{(k)(i)} + W_{jka}^i \alpha_{(l)(i)}] = 0.$$

We can, therefore, state the following theorem:

Theorem 3.3. *A necessary condition that the projective map of a PSK_n -space be a space of the same kind is that (3.13) holds.*

$$(3.14) \quad \text{Applying the commutation formula (1.6) to } W_{lm}^i, \text{ we have} \\ (\nabla_j \nabla_k' - \nabla_k' \nabla_j) W_{lm}^i = -B_{jkr}^{i\cdots i} W_{lm}^r + B_{jkl}^{i\cdots r} W_{rm}^i + B_{jkm}^{i\cdots r} W_{lr}^i - {}^r W_{lr}^i.$$

In view of equations (2.5), and (3.1), (3.14) becomes

In view of equations (2.5), and (3.1), (3.14) becomes

$$(3.15) \quad W_{rm}^i B_{jkl}^{i\cdots r} + W_{lr}^i B_{jkm}^{i\cdots r} - W_{lm}^r B_{jkr}^{i\cdots i} = 0.$$

Thus, we have

Theorem 3.4. *In a PSK_n -space, the relation (3.15) holds.*

The Bianchi identities for the projective curvature tensor in a special Kawaguchi space have been obtained by one of the present authors [6]. They are given by

$$\begin{aligned}
 (3.16) \quad & \nabla_m W_{jkl}^i + \nabla_j W_{kml}^i + \nabla_k W_{mjl}^i = \nabla_m R_{jkl}^{\cdot\cdot\cdot i} + \nabla_j R_{kml}^{\cdot\cdot\cdot i} + \\
 & + \nabla_k R_{mjl}^{\cdot\cdot\cdot i} - \frac{1}{n-1} [\delta_j^i (\nabla_m R_{akl}^{\cdot\cdot\cdot a} - \nabla_k R_{aml}^{\cdot\cdot\cdot a}) + \\
 & + \delta_k^i (\nabla_j R_{aml}^{\cdot\cdot\cdot a} - \nabla_m R_{ajl}^{\cdot\cdot\cdot a}) + \delta_m^i (\nabla_k R_{ajl}^{\cdot\cdot\cdot a} - \nabla_j R_{akl}^{\cdot\cdot\cdot a})] - \\
 & - \frac{x'^i}{n+1} [\nabla_l' (\nabla_m R_{jka}^{\cdot\cdot\cdot a} + \nabla_j R_{kma}^{\cdot\cdot\cdot a} + \nabla_k R_{mja}^{\cdot\cdot\cdot a})] + \\
 & + \frac{x'^h}{n^2-1} [\delta_j^i \nabla_l' (\nabla_m R_{hka}^{\cdot\cdot\cdot a} - \nabla_k R_{hma}^{\cdot\cdot\cdot a}) + \delta_k^i \nabla_l' (\nabla_j R_{hma}^{\cdot\cdot\cdot a} - \nabla_m R_{hja}^{\cdot\cdot\cdot a}) + \\
 & + \delta_m^i \nabla_l' (\nabla_k R_{hja}^{\cdot\cdot\cdot a} - \nabla_j R_{hka}^{\cdot\cdot\cdot a})] - \frac{1}{n^2-1} [\delta_j^i (\nabla_m R_{kla}^{\cdot\cdot\cdot a} - \nabla_k R_{mla}^{\cdot\cdot\cdot a}) + \\
 & + \delta_k^i (\nabla_j R_{mla}^{\cdot\cdot\cdot a} - \nabla_m R_{jla}^{\cdot\cdot\cdot a}) + \delta_m^i (\nabla_k R_{jla}^{\cdot\cdot\cdot a} - \nabla_j R_{kla}^{\cdot\cdot\cdot a})] - \\
 & - \frac{\delta_l^i}{n+1} (\nabla_m R_{jka}^{\cdot\cdot\cdot a} + \nabla_j R_{kma}^{\cdot\cdot\cdot a} + \nabla_k R_{mja}^{\cdot\cdot\cdot a}).
 \end{aligned}$$

Tansvecting (3.16) with x'^l and using (1.11), (2.5), (1.13) and the fact that $\nabla_m R_{jkl}^{\cdot\cdot\cdot i}$ is homogeneous of degree zero with respect to x'^i , we have

$$\begin{aligned}
 (3.17) \quad & \nabla_m W_{jk}^i + \nabla_j W_{km}^i + \nabla_k W_{mj}^i = -\frac{1}{n-1} [\delta_j^i (\nabla_m K_{ak}^{\cdot\cdot\cdot a} - \nabla_k K_{am}^{\cdot\cdot\cdot a}) + \\
 & + \delta_k^i (\nabla_j K_{am}^{\cdot\cdot\cdot a} - \nabla_m K_{aj}^{\cdot\cdot\cdot a}) + \delta_m^i (\nabla_k K_{aj}^{\cdot\cdot\cdot a} - \nabla_j K_{ak}^{\cdot\cdot\cdot a})] - \\
 & - \frac{x'^l}{n^2-1} [\delta_j^i (\nabla_m R_{kla}^{\cdot\cdot\cdot a} - \nabla_k R_{mla}^{\cdot\cdot\cdot a}) + \delta_k^i (\nabla_j R_{mla}^{\cdot\cdot\cdot a} - \nabla_m R_{jla}^{\cdot\cdot\cdot a}) + \\
 & + \delta_m^i (\nabla_k R_{jla}^{\cdot\cdot\cdot a} - \nabla_j R_{kla}^{\cdot\cdot\cdot a})] - \frac{x'^i}{n+1} (\nabla_m R_{jka}^{\cdot\cdot\cdot a} + \nabla_j R_{kma}^{\cdot\cdot\cdot a} + \nabla_k R_{mja}^{\cdot\cdot\cdot a}).
 \end{aligned}$$

For a PSK_n -space, the left hand side of (3.17) is zero and accordingly, we have

Theorem 3.5. *In a PSK_n -space, the relation*

$$\begin{aligned}
 (3.18) \quad & \frac{1}{n-1} [\delta_j^i (\nabla_m K_{ak}^{\cdot\cdot\cdot a} - \nabla_k K_{am}^{\cdot\cdot\cdot a}) + \delta_k^i (\nabla_j K_{am}^{\cdot\cdot\cdot a} - \nabla_m K_{aj}^{\cdot\cdot\cdot a}) + \delta_m^i (\nabla_k K_{aj}^{\cdot\cdot\cdot a} - \nabla_j K_{ak}^{\cdot\cdot\cdot a})] + \\
 & + \frac{x'^l}{n^2-1} [\delta_j^i (\nabla_m R_{kla}^{\cdot\cdot\cdot a} - \nabla_k R_{mla}^{\cdot\cdot\cdot a}) + \delta_k^i (\nabla_j R_{mla}^{\cdot\cdot\cdot a} - \nabla_m R_{jla}^{\cdot\cdot\cdot a}) + \delta_m^i (\nabla_k R_{jla}^{\cdot\cdot\cdot a} - \nabla_j R_{kla}^{\cdot\cdot\cdot a})] - \\
 & - \frac{x'^i}{n+1} (\nabla_m R_{jka}^{\cdot\cdot\cdot a} + \nabla_j R_{kma}^{\cdot\cdot\cdot a} + \nabla_k R_{mja}^{\cdot\cdot\cdot a}) = 0,
 \end{aligned}$$

holds.

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