

Generalized difference-property for functions of several variables

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0. Introduction

Setting out from N. G. DE BRUIJN's investigations (see [2]) numerous authors studied difference-properties (e.g. [2], [3], [4]). An *algebraic difference-property*, first to be found in the paper of Z. DARÓCZY—K. LAJKÓ—L. SZÉKELYHIDI [5], is the following:

Denote by P the set of positive elements of an ordered field and let A be an additive Abelian group. Let $f: P \rightarrow A$ be an arbitrary function and $\Delta_\lambda f: P \rightarrow A$ be the function defined by

$$(1) \quad \Delta_\lambda f(x) = f(\lambda x) - f(x) \quad (x \in P)$$

for all fixed $\lambda \in P$. The function $\alpha: P \rightarrow A$ is called a *Jensen-function* if

$$(2) \quad 2\alpha\left(\frac{x_1 + x_2}{2}\right) = \alpha(x_1) + \alpha(x_2) \quad (x_1, x_2 \in P)$$

holds. Denote by $J(P \rightarrow A)$ the class of Jensen-functions.

The next theorem is due to Z. DARÓCZY—K. LAJKÓ—L. SZÉKELYHIDI.

Theorem 0.1. *Let $f: P \rightarrow A$ be an arbitrary function, such that the function $\Delta_\lambda f$ defined by (1) is a Jensen-function for every fixed $\lambda \in P$. Then there exists a function $\alpha \in J(P \rightarrow A)$ such that the function $m(x) = f(x) - \alpha(x)$ ($x \in P$) is a homomorphism, i.e.*

$$(3) \quad m(x_1 x_2) = m(x_1) + m(x_2) \quad (x_1, x_2 \in P)$$

holds.

This result shows that the *Jensen-property* is a *difference-property*.

In this paper we are going to investigate a generalization of the above difference-property.

Our notations are the following: Denote by P^n the set

$$\underbrace{P \times P \times \dots \times P}_n \quad \text{and for}$$

$$\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in P^n, \quad \underline{x} = (x_1, \dots, x_n) \in P^n \quad \text{let} \quad \underline{\lambda} \underline{x} \doteq (\lambda_1 x_1, \dots, \lambda_n x_n).$$

If $\underline{x} \in P^n$ then $\underline{x}_i(t)$ will denote the vector

$$(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n).$$

Let $F: P^n \rightarrow A$ be an arbitrary function and $\Delta_{\underline{\lambda}} F: P^n \rightarrow A$ be the function defined by

$$(4) \quad \Delta_{\underline{\lambda}} F(\underline{x}) = F(\underline{\lambda}\underline{x}) - F(\underline{x}) \quad (\underline{x}, \underline{\lambda} \in P^n).$$

The function $H: P^n \rightarrow A$ is said to be a generalized Jensen-function, if

$$(5) \quad 2H\left(\underline{x}_i\left(\frac{t+s}{2}\right)\right) = H(\underline{x}_i(t)) + H(\underline{x}_i(s))$$

holds for every $i=1, \dots, n$; $\underline{x} \in P^n$ and $t, s \in P$. The class of generalized Jensen-functions will be denoted by $J(P^n \rightarrow A)$.

The property T is said to be a generalized difference-property if the following holds: for any $F: P^n \rightarrow A$, such that $\Delta_{\underline{\lambda}} F: P^n \rightarrow A$ has the property T for every $\underline{\lambda} \in P^n$ the representation

$$(6) \quad F(\underline{x}) = H^*(\underline{x}) + \sum_{i=1}^n m_i(x_i) \quad (\underline{x} \in P^n)$$

is valid, where H^* is a function of property T and $m_i: P \rightarrow A$ are homomorphisms of P into A ($i=1, \dots, n$).

We prove that the generalized Jensen-property is a generalized difference-property in this general sense. Besides in the case $\mathbf{R}_+ = \{x | x > 0\}$ we shall give the general form of generalized Jensen-functions.

1. A generalized difference-property

The generalization of theorem 0.1. can be formulated as follows:

Theorem 1.1. Let $F: P^n \rightarrow A$ be such a function that $\Delta_{\underline{\lambda}} F \in J(P^n \rightarrow A)$ for all $\underline{\lambda} \in P^n$, then □

$$(7) \quad F(\underline{x}) = \alpha(\underline{x}) + \sum_{i=1}^n m_i(x_i) \quad (\underline{x} \in P^n),$$

where $\alpha \in J(P^n \rightarrow A)$ and $m_i: P \rightarrow A$ ($i=1, \dots, n$) are homomorphisms of P into A .

PROOF. Our method is induction on n , the number of variables of F .

a) If $n=1$, the proposition is just the theorem 0.1. which was proved in [5].

b) Let us suppose that the theorem holds for functions of n variables, then we show that it is true for functions of $n+1$ variables.

We shall use the notations

$$(\underline{x}, x_{n+1}) = (x_1, \dots, x_n, x_{n+1}),$$

$$(\underline{\lambda}, \lambda_{n+1}) = (\lambda_1, \dots, \lambda_n, \lambda_{n+1})$$

for elements of P^{n+1} .

Assume that $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F \in J(P^{n+1} \rightarrow A)$.

(i) First we demonstrate that the function defined by

$$(9) \quad \varphi(\underline{x}, x_{n+1}) = F(\underline{x}, 2x_{n+1}) - F(\underline{x}, x_{n+1}), \quad ((\underline{x}, x_{n+1}) \in P^{n+1})$$

has the form

$$(10) \quad \varphi(\underline{x}, x_{n+1}) = \varphi(\underline{x}, 1) + m_{n+1}(x_{n+1}) - \varphi(\underline{1}, 1), \quad ((\underline{x}, x_{n+1}) \in P^{n+1}).$$

Namely if $\beta \in J(P^{n+1} \rightarrow A)$, then

$$2\beta\left(\underline{x}, \frac{t_1+t_2}{2}\right) = \beta(\underline{x}, t_1) + \beta(\underline{x}, t_2) \quad (\underline{x} \in P^n; t_1, t_2 \in P).$$

Let us substitute here in order $t_1 = 2x_{n+1}$, $t_2 = 2$;

$$t_1 = x_{n+1} + 1, \quad t_2 = 1; \quad t_1 = x_{n+1}, \quad t_2 = 2,$$

then we obtain the equations

$$2\beta(\underline{x}, x_{n+1} + 1) = \beta(\underline{x}, 2x_{n+1}) + \beta(\underline{x}, 2),$$

$$2\beta\left(\underline{x}, \frac{x_{n+1}+2}{2}\right) = \beta(\underline{x}, x_{n+1} + 1) + \beta(\underline{x}, 1),$$

$$2\beta\left(\underline{x}, \frac{x_{n+1}+2}{2}\right) = \beta(\underline{x}, x_{n+1}) + \beta(\underline{x}, 2)$$

for all $\underline{x} \in P^n$, $x_{n+1} \in P$. Comparing these equations we obtain that β satisfies the functional equation

$$(11) \quad \beta(\underline{x}, 2x_{n+1}) - 2\beta(\underline{x}, x_{n+1}) = \beta(\underline{x}, 2) - 2\beta(\underline{x}, 1), \quad (\underline{x} \in P^n, x_{n+1} \in P).$$

By $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F \in J(P^n \rightarrow A)$ it follows from (11) that

$$(12) \quad \begin{cases} \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2x_{n+1}) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, x_{n+1}) = \\ = \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 1) = \\ = F(\underline{\lambda}\underline{x}, 2\lambda_{n+1}) - F(\underline{x}, 2) - 2F(\underline{\lambda}\underline{x}, \lambda_{n+1}) + 2F(\underline{x}, 1) \end{cases}$$

is valid for all $(\underline{x}, x_{n+1}), (\underline{\lambda}, \lambda_{n+1}) \in P^{n+1}$.

On the other hand from the definition of the function $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F$ we get

$$(13) \quad \begin{cases} \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2x_{n+1}) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, \lambda_{n+1}) = \\ = F(\underline{\lambda}\underline{x}, 2\lambda_{n+1}x_{n+1}) - F(\underline{x}, 2x_{n+1}) - 2F(\underline{\lambda}\underline{x}, \lambda_{n+1}x_{n+1}) + 2F(\underline{x}, x_{n+1}). \end{cases}$$

Equations (12) and (13) immediately imply that the function $\varphi: P^{n+1} \rightarrow A$ defined by (9) satisfies the functional equation

$$(14) \quad \varphi(\underline{\lambda}\underline{x}, \lambda_{n+1}x_{n+1}) - \varphi(\underline{x}, x_{n+1}) = \varphi(\underline{\lambda}\underline{x}, \lambda_{n+1}) - \varphi(\underline{x}, 1)$$

for all $(\underline{x}, x_{n+1}), (\underline{\lambda}, \lambda_{n+1}) \in P^{n+1}$. From (14) by substitution $\lambda_{n+1} = 1$, $\underline{x} = \underline{1}$ we have

$$(15) \quad \varphi(\underline{\lambda}, x_{n+1}) = \varphi(\underline{\lambda}, 1) + \varphi(\underline{1}, x_{n+1}) - \varphi(\underline{1}, 1) \quad (\underline{\lambda} \in P^n, x_{n+1} \in P).$$

Substituting (15) in (14) we easily obtain that

$$\varphi(\underline{1}, \lambda_{n+1}x_{n+1}) = \varphi(\underline{1}, \lambda_{n+1}) + \varphi(\underline{1}, x_{n+1})$$

for all $\lambda_{n+1}, x_{n+1} \in P$, i.e. the function $m_{n+1}: P \rightarrow A$ defined by

$$m_{n+1}(x_{n+1}) = \varphi(\underline{1}, x_{n+1}) \quad (x_{n+1} \in P)$$

is a homomorphism.

Thus (15) shows that φ is of form (10).

(ii) *The definition of function φ implies that*

$$\varphi(\lambda \underline{x}, 1) - \varphi(\underline{x}, 1) = \Delta_{(\lambda, 1)} F(\underline{x}, 2) - 2\Delta_{(\lambda, 1)} F(\underline{x}, 1),$$

i.e. $\varphi(\lambda \underline{x}, 1) - \varphi(\underline{x}, 1) \in J(P^n \rightarrow A)$ for all $\lambda \in P^n$. By our assumption b) there is a function $\bar{\alpha} \in J(P^n \rightarrow A)$ and there are homomorphisms $\bar{m}_i: P \rightarrow A$ ($i=1, 2, \dots, n$), such that

$$(16) \quad \varphi(\underline{x}, 1) = \bar{\alpha}(\underline{x}) + \sum_{i=1}^n \bar{m}_i(x_i) \quad (\underline{x} \in P^n).$$

(iii) $\Delta_{(\lambda, \lambda_{n+1})} F \in J(P^{n+1} \rightarrow A)$ implies that the function $\alpha_1: P^{n+1} \rightarrow A$ defined by

$$(17) \quad \alpha_1(\underline{x}, x_{n+1}) = F(\underline{x}, 2x_{n+1}) - F(\underline{x}, x_{n+1}) \quad ((\underline{x}, x_{n+1}) \in P^{n+1})$$

belongs to $J(P^{n+1} \rightarrow A)$.

On the other hand from (17) and (10) it follows also

$$(18) \quad \alpha_1(\underline{x}, x_{n+1}) = \varphi(\underline{x}, x_{n+1}) + F(\underline{x}, x_{n+1}) \quad ((\underline{x}, x_{n+1}) \in P^{n+1}).$$

(iv) *Using (18), (10) and (16) we obtain that*

$$F(\underline{x}, x_{n+1}) = \alpha_1(\underline{x}, x_{n+1}) - \bar{\alpha}(\underline{x}) - \sum_{i=1}^{n+1} \bar{m}_i(x_i) - \varphi(\underline{1}, 1)$$

for all $(\underline{x}, x_{n+1}) \in P^{n+1}$.

Let α be the function defined by

$$\alpha(\underline{x}, x_{n+1}) = \alpha_1(\underline{x}, x_{n+1}) - \bar{\alpha}(\underline{x}) - \varphi(\underline{1}, 1) \quad ((\underline{x}, x_{n+1}) \in P^{n+1}),$$

then $\alpha \in J(P^{n+1} \rightarrow A)$ and by the notation

$m_i(x_i) = -\bar{m}_i(x_i)$ we obtain that

$$F(\underline{x}, x_{n+1}) = \alpha(\underline{x}, x_{n+1}) + \sum_{i=1}^{n+1} m_i(x_i) \quad ((\underline{x}, x_{n+1}) \in P^{n+1}),$$

which corresponds to the form (8).

This completes the proof of theorem 1.1.

2. The general form of generalized Jensen-functions

The formula (7) given in the theorem 1.1. set the problem of the representation of generalized Jensen-functions. We shall demonstrate that if $P^n = \mathbf{R}_+^n$ (where $\mathbf{R}_+$ is the set of positive real numbers) and $A = \mathbf{R}$, then there is a close connection between the Jensen-functions of $J(\mathbf{R}_+^n \rightarrow \mathbf{R})$ and multiadditive functions. In this case we find the general form of Jensen-functions by the help of multiadditive functions.

To prove our theorem we need the following result of [5]:

Lemma. *If the function $\alpha: \mathbf{R}_+ \rightarrow \mathbf{R}$ belongs to $J(\mathbf{R}_+ \rightarrow \mathbf{R})$, then α has the form*

$$(19) \quad \alpha(x) = A(x) + b \quad (x \in \mathbf{R}_+),$$

where the function $A: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the Cauchy functional equation

$$(20) \quad A(x+y) = A(x) + A(y) \quad ((x, y) \in \mathbf{R}^2)$$

(i.e. additive) and $b \in \mathbf{R}$ is an arbitrary constant.

Using induction on n , the number of variables, we shall prove

Theorem 2.1. *If the function $\alpha: \mathbf{R}_+^n \rightarrow \mathbf{R}$ belongs to $J(\mathbf{R}_+^n \rightarrow \mathbf{R})$, then α has the form*

$$(21) \quad \begin{cases} \alpha(\underline{x}) = A_n^0(x_1, \dots, x_n) + A_{n-1}^1(x_1, \dots, x_{n-1}) + \dots + A_{n-1}^n(x_2, \dots, x_n) + \\ + A_{n-2}^1(x_1, \dots, x_{n-2}) + \dots + A_{n-2}^{\frac{n(n-1)}{2}}(x_3, \dots, x_n) + \dots + A_2^1(x_1, x_2) + \\ + \dots + A_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n) + A_1^1(x_1) + \dots + A_1^n(x_n) + A_0 \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$ where the functions $A_i^k: \mathbf{R}^i \rightarrow \mathbf{R}$ ($i=1, \dots, n$; $k=0, \dots, \binom{n}{i}$; $i=1, \dots, n-1$) are additive in each variable and $A_0 \in \mathbf{R}$ is an arbitrary constant.

PROOF. a) If $n=1$, then the statement is true by the lemma.

b) Let us suppose that our statement holds for functions with n variables, then we prove that it is satisfied for $n+1$ variables too.

If α belongs to $J(\mathbf{R}_+^{n+1} \rightarrow \mathbf{R})$, then using the Jensen-property of α in the $(n+1)$ th variable (because of our lemma) α has the form

$$(22) \quad \alpha(\underline{x}, x_{n+1}) = \bar{A}(\underline{x}, x_{n+1}) + b(\underline{x})$$

for all $\underline{x} \in \mathbf{R}_+^n$, $x_{n+1} \in \mathbf{R}_+$, where the function $\bar{A}: \mathbf{R}_+^n \times \mathbf{R} \rightarrow \mathbf{R}$ is additive in the $(n+1)$ th variable and $b: \mathbf{R}_+^n \rightarrow \mathbf{R}$ is an arbitrary function. Since α is a Jensen-function in each variable, using (22) we get the equation

$$(23) \quad \begin{cases} 2\bar{A}\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right), x_{n+1}\right) + 2b\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right)\right) = \bar{A}(\underline{x}_i(t_1), x_{n+1}) + \\ + \bar{A}(\underline{x}_i(t_2), x_{n+1}) + b(\underline{x}_i(t_1)) + b(\underline{x}_i(t_2)) \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$, $x_{n+1} \in \mathbf{R}$; $t_1, t_2 \in \mathbf{R}_+$ and $i=1, \dots, n$.

The additivity of $\bar{A}$ in the $(n+1)$ th variable and (23) imply that

$$\begin{aligned} x_{n+1} \left[2\bar{A}\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right), 1\right) - \bar{A}(\underline{x}_i(t_1), 1) - \bar{A}(\underline{x}_i(t_2), 1) \right] = \\ = b(\underline{x}_i(t_1)) + b(\underline{x}_i(t_2)) - 2b\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right)\right) \end{aligned}$$

is valid for all positive rational x_{n+1} and $i=1, \dots, n$; $\underline{x} \in \mathbf{R}_+^n$; $t_1, t_2 \in \mathbf{R}_+$. This is possible if and only if the coefficient of x_{n+1} and the right hand side of this equation

is equal to zero for all $\underline{x} \in \mathbf{R}_+^n$, $t_1, t_2 \in \mathbf{R}_+$ and $i=1, \dots, n$, which shows that the function b belongs to $J(\mathbf{R}_+^n \rightarrow \mathbf{R})$.

By induction hypothesis it follows then that b has the form

$$(24) \quad \begin{cases} b(\underline{x}) = B_n^0(\underline{x}) + B_{n-1}^1(x_1, \dots, x_{n-1}) + \dots + B_{n-1}^n(x_2, \dots, x_n) + \\ + B_{n-2}^1(x_1, \dots, x_{n-2}) + \dots + B_{n-2}^{\frac{n(n-1)}{2}}(x_3, \dots, x_n) + \dots + B_2^1(x_1, x_2) + \\ + \dots + B_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n) + B_1^1(x_1) + \dots + B_1^n(x_n) + B_0 \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$, where the functions $B_i^k: \mathbf{R}^i \rightarrow \mathbf{R}$ are additive in each variable and $B_0 \in \mathbf{R}$ is an arbitrary constant.

On the other hand $b \in J(\mathbf{R}_+^n \rightarrow \mathbf{R})$ and (23) imply that the function $\bar{A}$ has the Jensen-property in each of its first n variables, which means that for every fixed $x_{n+1} \in \mathbf{R}_+$ moreover for $x_{n+1} \in \mathbf{R}$ $\bar{A}$ belongs to $J(\mathbf{R}_+^n \rightarrow \mathbf{R})$. Thus by the induction hypothesis we have

$$(25) \quad \begin{cases} \bar{A}(\underline{x}, x_{n+1}) = \tilde{A}_n^0(\underline{x}, x_{n+1}) + \tilde{A}_{n-1}^1(x_1, \dots, x_{n-1}, x_{n+1}) + \dots + \\ + \tilde{A}_{n-1}^n(x_2, \dots, x_n, x_{n+1}) + \dots + \tilde{A}_1^1(x_1, x_{n+1}) + \dots + \\ + \tilde{A}_1^n(x_n, x_{n+1}) + \tilde{A}_0(x_{n+1}) \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$, $x_{n+1} \in \mathbf{R}$ where the functions $\tilde{A}_i^k: \mathbf{R}^i \rightarrow \mathbf{R}$ are additive in each variable $i \leq n$, and $\tilde{A}_0: \mathbf{R} \rightarrow \mathbf{R}$ is an arbitrary function.

To complete the proof it is sufficient to show that every function $\tilde{A}_i^k$ is additive in the last variable too.

The function $\bar{A}$ is additive in the last variable, therefore from (25) we obtain the functional equation

$$(26) \quad \begin{cases} \tilde{A}_n^0(x_1, \dots, x_n, t_1+t_2) + \tilde{A}_{n-1}^1(x_1, \dots, x_{n-1}, t_1+t_2) + \dots + \\ + \tilde{A}_{n-1}^n(x_2, \dots, x_n, t_1+t_2) + \dots + \tilde{A}_1^1(x_1, t_1+t_2) + \dots + \tilde{A}_1^n(x_n, t_1+t_2) + \\ + \tilde{A}_0(t_1+t_2) = \tilde{A}_n^0(x_1, \dots, x_n, t_1) + \tilde{A}_n^0(x_1, \dots, x_n, t_2) + \\ + \tilde{A}_{n-1}^1(x_1, \dots, x_{n-1}, t_1) + \tilde{A}_{n-1}^1(x_1, \dots, x_{n-1}, t_2) + \dots + \\ + \tilde{A}_{n-1}^n(x_2, \dots, x_n, t_1) + \tilde{A}_{n-1}^n(x_2, \dots, x_n, t_2) + \dots + \tilde{A}_1^1(x_1, t_1) + \\ + \tilde{A}_1^1(x_1, t_2) + \dots + \tilde{A}_1^n(x_n, t_1) + \tilde{A}_1^n(x_n, t_2) + \tilde{A}_0(t_1) + \tilde{A}_0(t_2) \end{cases}$$

for every fixed $x_i \in \mathbf{R}_+$ ($i=1, \dots, n$) and for all $t_1, t_2 \in \mathbf{R}$. Using the multiadditivity we obtain from (26) that

$$\begin{aligned} & x_1 \cdot \dots \cdot x_n [\tilde{A}_n^0(\underline{1}, t_1+t_2) - \tilde{A}_n^0(\underline{1}, t_1) - \tilde{A}_n^0(\underline{1}, t_2)] + \\ & + x_1 \cdot \dots \cdot x_{n-1} [\tilde{A}_{n-1}^1(\underline{1}, t_1+t_2) - \tilde{A}_{n-1}^1(\underline{1}, t_1) - \tilde{A}_{n-1}^1(\underline{1}, t_2)] + \\ & + \dots + x_2 \cdot \dots \cdot x_n [\tilde{A}_{n-1}^n(\underline{1}, t_1+t_2) - \tilde{A}_{n-1}^n(\underline{1}, t_1) - \tilde{A}_{n-1}^n(\underline{1}, t_2)] + \\ & + \dots + x_1 [\tilde{A}_1^1(1, t_1+t_2) - \tilde{A}_1^1(1, t_1) - \tilde{A}_1^1(1, t_2)] + \\ & + \dots + x_n [\tilde{A}_1^n(1, t_1+t_2) - \tilde{A}_1^n(1, t_1) - \tilde{A}_1^n(1, t_2)] = \tilde{A}_0(t_1) + \tilde{A}_0(t_2) - \tilde{A}_0(t_1+t_2) \end{aligned}$$

for arbitrary positive rational numbers $x_1, \dots, x_n$ and for all $t_1, t_2 \in \mathbf{R}$. This is possible if and only if the coefficients in the brackets and the right hand side of this equation is equal to zero for all $t_1, t_2 \in \mathbf{R}$. This means that the function $\tilde{A}_0: \mathbf{R} \rightarrow \mathbf{R}$ is additive on the whole plane.

If all x'_j 's ($j=1, \dots, n$) are rational except the i th ($i=1, \dots, n$) one then using a similar reasoning then before we get from (26) that

$$\begin{aligned} & x_1 \cdot \dots \cdot x_{i-1} \cdot x_{i+1} \cdot \dots \cdot x_n [\tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, 1, t_1+t_2) - \tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, 1, t_1) - \\ & - \tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, 1, t_2)] + \dots + x_1 \cdot \dots \cdot x_{i-1} x_{i+1} \cdot \dots \cdot x_{n-1} [\quad] + \\ & + \dots + x_2 \cdot \dots \cdot x_{i-1} \cdot x_{i+1} \cdot \dots \cdot x_n [\tilde{A}_{n-1}^n(1, \dots, x_i, \dots, 1, t_1+t_2) - \\ & - \tilde{A}_{n-1}^n(1, \dots, x_i, \dots, 1, t_1) - \tilde{A}_{n-1}^n(1, \dots, x_i, \dots, 1, t_2)] + \dots + \\ & + x_1 [\tilde{A}_1^1(1, t_1+t_2) - \tilde{A}_1^1(1, t_1) - \tilde{A}_1^1(1, t_2)] + \dots + [\tilde{A}_1^i(x_i, t_1+t_2) - \tilde{A}_1^i(x_i, t_1) - \\ & - \tilde{A}_1^i(x_i, t_2)] + \dots + x_n [\tilde{A}_1^n(1, t_1+t_2) - \tilde{A}_1^n(1, t_1) - \tilde{A}_1^n(1, t_2)] = 0 \end{aligned}$$

for $x_i \in \mathbf{R}_+$ ($i=1, \dots, n$), $t_1, t_2 \in \mathbf{R}$. This implies that the functions $\tilde{A}_1^i: \mathbf{R}^2 \rightarrow \mathbf{R}$ ($i=1, \dots, n$) are additive in the 2th variable for $x_i \in \mathbf{R}$ and by the extensions:

$$\tilde{A}_1^i(x_i, t) = -\tilde{A}_1^i(-x_i, t) \quad (x_i < 0); \quad \tilde{A}_1^i(0, t) = 0$$

this follows for all $x_i \in \mathbf{R}$ too and then the functions $\tilde{A}_1^i: \mathbf{R} \rightarrow \mathbf{R}$ are biadditive.

Let now $x_1, \dots, x_n$ be rational numbers except x_i and x_j , then from (26) we obtain that the functions $\tilde{A}_2^i: \mathbf{R}^3 \rightarrow \mathbf{R}$ ($i=1, \dots, \binom{n}{2}$) are additive in each variable.

Continuing similarly we obtain that the functions $\tilde{A}_3^i: \mathbf{R}^4 \rightarrow \mathbf{R}, \dots, \tilde{A}_{n-1}^i: \mathbf{R}^n \rightarrow \mathbf{R}$ and finally $\tilde{A}_n^i: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$ are additive in each variable, i.e. they are multiadditive functions.

From (22), (24) and (25) we obtain (21) for $n+1$ variables.

Thus the proof is complete.

3. The investigation of generalized difference-property in the case

$$P^n = \mathbf{R}_+^n, \quad A = \mathbf{R}$$

Knowing the general form of generalized Jensen-functions in the case $P^n = \mathbf{R}_+^n$ and $A = \mathbf{R}$, the theorem 1.1 can be formulated as follows:

Theorem 3.1. Let $F: \mathbf{R}_+^n \rightarrow \mathbf{R}$ be a function, such that the function $\Delta_{\lambda} F: \mathbf{R}_+^n \rightarrow \mathbf{R}$ defined by (4) belongs to $J(\mathbf{R}_+^n \rightarrow \mathbf{R})$ for all $\lambda \in \mathbf{R}_+^n$. Then there exist homomorphisms $m_i: \mathbf{R}_+ \rightarrow \mathbf{R}$ ($i=1, \dots, n$), multiadditive functions $A_i^k: \mathbf{R}^i \rightarrow \mathbf{R}$ ($i=1, \dots, n; k=$

$=0, \dots, \binom{n}{l}; l=1, \dots, n-1$) and $A_0 \in \mathbf{R}$ constant, such that

$$(27) \quad \begin{cases} F(x_1, \dots, x_n) = A_n^0(x_1, \dots, x_n) + A_{n-1}^1(x_1, \dots, x_{n-1}) + \dots + \\ + A_{n-1}^n(x_2, \dots, x_n) + \dots + A_2^1(x_1, x_2) + \dots + A_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n) + \\ + A_1^1(x_1) + \dots + A_1^n(x_n) + \sum_{i=1}^n m_i(x_i) + A_0 \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$.

PROOF. The statement of this theorem immediately follows from theorems 1.1. and 2.1. i.e. from the formulae (7) and (21).

In the case $n=2$ theorem 3.1. can be applied to determine the general solution of "rectangle-type" functional equations (see [6]), therefore it is interesting to reformulate our theorem in this case:

Corollary. Let $F: \mathbf{R}_+^2 \rightarrow \mathbf{R}$ be a function, such that the function $\Delta_{\lambda, \mu} F: \mathbf{R}_+^2 \rightarrow \mathbf{R}$ defined by $\Delta_{\lambda, \mu} F(x, y) = F(\lambda x, \mu y) - F(x, y)$ ($x, y, \lambda, \mu \in \mathbf{R}_+$) belongs to $J(\mathbf{R}_+^2 \rightarrow \mathbf{R})$ for all $(\lambda, \mu) \in \mathbf{R}_+^2$. Then there exist homomorphisms $m_1, m_2: \mathbf{R}_+ \rightarrow \mathbf{R}$, a biadditive function $A: \mathbf{R}^2 \rightarrow \mathbf{R}$, additive functions $a_1, a_2: \mathbf{R} \rightarrow \mathbf{R}$ and a constant $b \in \mathbf{R}$, such that

$$F(x, y) = A(x, y) + a_1(x) + a_2(y) + m_1(x) + m_2(y) + b \quad (x, y \in \mathbf{R}_+).$$

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