

Non-additive Ring and Module Theory, III. Morita Equivalences

Dedicated to the memory of Andor Kertész

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This paper is a continuation of [19], [20]. References are quoted there.

As in ring theory one may ask the question when two categories ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ for monoids A and B are equivalent. Now in ring theory we know from the additivity of the equivalences $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ that the natural bijections

$$\mathrm{Hom}_B(\mathcal{F}(M), N) \cong \mathrm{Hom}_A(M, \mathcal{G}(N)) \quad \text{and} \quad \mathrm{Hom}_A(\mathcal{G}(N), M) \cong \mathrm{Hom}_B(N, \mathcal{F}(M))$$

are isomorphisms of abelian groups. So in the general case we only want to consider equivalences such that there are isomorphisms ${}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)]$ and ${}_A[\mathcal{G}(N), M] \cong {}_B[N, \mathcal{F}(M)]$. In view of theorem 4.3. this is equivalent to studying equivalences such that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}$ -functors. The last condition can be studied even in monoidal, non-closed categories.

We call ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ $\mathcal{C}$ -equivalent if there are inverse equivalences $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ such that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}$ -functors.

Without loss of generality we shall only consider equivalences $\mathcal{F}$ and $\mathcal{G}$ together with isomorphisms $\Phi: \mathcal{F}\mathcal{G} \cong \mathrm{Id}$ and $\Psi: \mathcal{G}\mathcal{F} \cong \mathrm{Id}$ such that $\mathcal{F}\Psi = \Phi\mathcal{F}$ and $\Psi\mathcal{G} = \mathcal{G}\Phi$. Then Φ and Ψ and their inverses are already adjointness morphisms.

5.1. Theorem. *Let $\mathcal{C}$ be an arbitrary monoidal category. Let $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ and $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ be inverse $\mathcal{C}$ -equivalences. Then there are objects $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_A$ such that*

a) *there are natural isomorphisms*

$$\mathcal{F}(M) \cong Q \otimes_A M \cong {}_A[P, M] \quad \text{in } {}_A\mathcal{C},$$

$$\mathcal{G}(N) \cong P \otimes_B N \cong {}_B[Q, N] \quad \text{in } {}_B\mathcal{C},$$

and P B -coflat and Q A -coflat.

b) *there are isomorphisms of A - A -resp. B - B -biobjects*

$$A \cong P \otimes_B Q \quad \text{and} \quad B \cong Q \otimes_A P$$

such that the diagrams

$$\begin{array}{ccc} P \otimes (Q \otimes P) \cong (P \otimes Q) \otimes P & \longrightarrow & A \otimes P \\ \downarrow & & \downarrow \\ P \otimes B & \longrightarrow & P \end{array}$$

and

$$\begin{array}{ccc} Q \otimes (P \otimes Q) \cong (Q \otimes P) \otimes Q & \longrightarrow & B \otimes Q \\ \downarrow & & \downarrow \\ Q \otimes A & \longrightarrow & Q \end{array}$$

commute,

c) there are isomorphisms

$${}_B[Q, B] \cong P \quad \text{in } {}_A\mathcal{C}_B,$$

$${}_A[P, A] \cong Q \quad \text{in } {}_B\mathcal{C}_A,$$

d) there are isomorphisms

$${}_B[Q, Q] \cong A \quad \text{in } {}_A\mathcal{C}_A \quad \text{and as monoids,}$$

$${}_A[P, P] \cong B \quad \text{in } {}_B\mathcal{C}_B \quad \text{and as monoids.}$$

PROOF. By the symmetry of the situation we only have to prove one half of the assertions.

There is an isomorphism $\mathcal{F}(M) \cong Q \otimes_A M$ natural in M by Theorem 4.2 since $\mathcal{F}$ is a $\mathcal{C}$ -functor and clearly preserves difference cokernels as an equivalence. By the same theorem we have $\mathcal{G}(N) \cong {}_B[Q, N]$ natural in N , since $\mathcal{G}$ is adjoint to $\mathcal{F}$. This proves a).

We have an isomorphism $A \cong \mathcal{G}\mathcal{F}(A) \cong P \otimes_B (Q \otimes_A A) \cong P \otimes_B Q$ in ${}_A\mathcal{C}$. Furthermore we have a commutative diagram

$$\begin{array}{ccccc} A \otimes A \cong P \otimes_B (Q \otimes_A (A \otimes A)) & \cong & P \otimes_B (Q \otimes_A A) \\ \downarrow & & \downarrow & & \downarrow \\ A & \cong & P \otimes_B (Q \otimes_A A) & \cong & P \otimes_B Q \end{array}$$

hence $A \cong P \otimes_B Q$ as A - A -biobjects.

The adjunction morphism $\Psi: \mathcal{G}\mathcal{F} \cong Id$ induces the evaluation morphism $\Psi': P \otimes_B {}_A[P, M] \cong M$ with $\Psi'(p \otimes f) = \langle p \rangle f$. By definition of the isomorphism $A \cong P \otimes_B Q$ we get a commutative diagram

$$\begin{array}{ccc} P \otimes_B (Q \otimes_A M) \cong (P \otimes_B Q) \otimes_A M \cong A \otimes_A M & & \\ \searrow & & \swarrow \\ P \otimes_B {}_A[P, M] & \xrightarrow{\Psi'} & M \end{array}$$

Hence if the isomorphism $P \otimes_B Q \cong A$ is described by $p \otimes_B q \mapsto pq$ and the morphism $Q \otimes_A M \mapsto {}_A[P, M]$ is given by

$$q \otimes_A m \mapsto \varphi(q \otimes_A m), \quad \text{we get} \quad (pq)m = \langle p \rangle \varphi(q \otimes_A m).$$

Now if $M \in {}_A\mathcal{C}_B$ then we get $\langle p \rangle \varphi(q \otimes_A mb) = (pq)(mb) = ((pq)m)b = (\langle p \rangle \varphi(q \otimes_A m))b = \langle p \rangle (\varphi(q \otimes_A m)b)$, hence $Q \otimes_A M$ and ${}_A[P, M]$ are isomorphic as B - B -biobjects.

In particular we get ${}_A[P, A] \cong Q$ as B - A -biobjects and ${}_A[P, P] \cong Q \otimes_A P \cong B$ as B - B -biobjects.

To prove the monoid isomorphisms we first observe that

$$\begin{array}{ccc} P \otimes (Q \otimes P) \cong (P \otimes Q) \otimes P & \longrightarrow & A \otimes P \quad \text{and} \\ \downarrow & & \downarrow \\ P \otimes B & \longrightarrow & P \\ Q \otimes (P \otimes Q) \cong (Q \otimes P) \otimes Q & \longrightarrow & B \otimes Q \\ \downarrow & & \downarrow \\ Q \otimes A & \longrightarrow & Q \end{array}$$

commute. This follows from $\mathcal{F} \cong Q \otimes_A$, $\mathcal{G} \cong P \otimes_B$ and from the fact that $\Phi\mathcal{F}: \mathcal{F}\mathcal{G}\mathcal{F} \rightarrow \mathcal{F}$ and $\mathcal{F}\Psi: \mathcal{F}\mathcal{G}\mathcal{F} \rightarrow \mathcal{F}$ resp. $\mathcal{G}\Phi: \mathcal{G}\mathcal{F}\mathcal{G} \rightarrow \mathcal{G}$ and $\Psi\mathcal{G}: \mathcal{G}\mathcal{F}\mathcal{G} \rightarrow \mathcal{G}$ are equal. So we get

$$\langle p \rangle \varphi(q' \otimes p') \varphi(q'' \otimes p'') = (pq')(p'q'')p'' = p(q'p')(q''p'').$$

Since the isomorphism ${}_A[P, P] \cong B$ is given by

$${}_A[P, P] \xleftarrow{\varphi} Q \otimes_A P \cong B$$

or $\varphi(q' \otimes p') \mapsto q'p'$, the composition $\varphi(q' \otimes p') \varphi(q'' \otimes p'')$ is mapped to the product $(q'p')(q''p'')$. If $\varphi(q' \otimes p')$ is the identity then $p(q'p') = p$ for all p . But ${}_A[P, P] \rightarrow B$ is an isomorphism hence $q'p' = 1 \in B$.

5.2. Corollary: The morphisms

$$P(X) \times {}_A[P, A](Y) \ni (p, f) \mapsto \langle p \rangle f \in A(X \otimes Y)$$

and

$${}_A[P, A](X) \times P(Y) \ni (f, p) \mapsto fp \in {}_A[P, P](X \otimes Y)$$

with $\langle p' \rangle (fp) := (\langle p' \rangle f)p$ induce isomorphisms

$$P \otimes_B {}_A[P, A] \cong A \quad \text{and} \quad {}_A[P, A] \otimes_A P \cong {}_A[P, P].$$

The analogous assertions hold for Q and B .

PROOF. The first isomorphism, the evaluation morphism, was discussed in the proof of 5.1. The second isomorphism is just given by

$${}_A[P, A] \otimes_A P \xrightarrow{\varphi \otimes P} Q \otimes_A P \xrightarrow{\varphi} {}_A[P, P].$$

We have seen that each $\mathcal{C}$ -equivalence is induced by some object $P \in {}_A\mathcal{C}_B$ with the properties of Corollary 5.2. The converse will be proved after a more detailed study of the properties exhibited in Corollary 5.2.

An object $P \in {}_A\mathcal{C}$ will be called *finite*, if ${}_A[P, A]$ and $B := {}_A[P, P]$ exist, if P is B -coflat and ${}_A[P, A]$ is A -coflat and if the morphism ${}_A[P, A] \otimes_A P \rightarrow {}_A[P, P]$ induced by ${}_A[P, A](X) \times P(Y) \ni (f, p) \mapsto fp \in {}_A[P, P](X \otimes Y)$ with $\langle p' \rangle fp = (\langle p' \rangle f)p$ is an isomorphism. P will be called *faithfully projective* if it is finite and if the morphism $P \otimes_B {}_A[P, A] \rightarrow A$ induced by the evaluation is also an isomorphism.

5.3. Theorem. Let A, B be monoids in $\mathcal{C}$, $P \in {}_A\mathcal{C}_B$ B -coflat and $Q \in {}_B\mathcal{C}_A$ A -coflat. Given morphisms $f: P \otimes_B Q \rightarrow A$ in ${}_A\mathcal{C}_A$ and $g: Q \otimes_A P \rightarrow B$ in ${}_B\mathcal{C}_B$ such that the diagrams

$$\begin{array}{ccc} P \otimes_B (Q \otimes_A P) \cong (P \otimes_B Q) \otimes_A P & \xrightarrow{f \otimes_A P} & A \otimes_A P \\ \downarrow P \otimes_B g & & \downarrow \\ P \otimes_B B & \xrightarrow{\quad} & P \\ Q \otimes_A (P \otimes_B Q) \cong (Q \otimes_A P) \otimes_B Q & \xrightarrow{g \otimes_B Q} & B \otimes_B Q \\ \downarrow Q \otimes_A f & & \downarrow \\ Q \otimes_A A & \xrightarrow{\quad} & Q \end{array}$$

commute. Assume that there is $p_0 \otimes_B q_0 \in P \otimes_B Q(I)$ such that $p_0 q_0 := f(p_0 \otimes_B q_0) = 1 \in A(I)$. Then f is an isomorphism. Assume that in addition there is $q_1 \otimes_A p_1 \in Q \otimes_A P(I)$ such that $q_1 p_1 := g(q_1 \otimes_A p_1) = 1 \in B(I)$. Then $P \otimes_B -: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ and $Q \otimes_A -: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ are inverse $\mathcal{C}$ -equivalences. In particular $P \in {}_A\mathcal{C}$ and $Q \in {}_B\mathcal{C}$ are faithfully projective.

PROOF. Define $f': A \rightarrow P \otimes_B Q$ by $f'(a) = ap_0 \otimes_B q_0$. Then $ff'(a) = ap_0 q_0 = a$ and $f'f(p \otimes_B q) = (pq)p_0 \otimes_B q_0 = p(qp_0) \otimes_B q_0 = p \otimes_B (qp_0)q_0 = p \otimes_B q(p_0 q_0) = p \otimes_B q$. Hence f is an isomorphism.

Furthermore the functors $P \otimes_B Q \otimes_A \cong A \otimes_A$ and $Q \otimes_A P \otimes_B \cong B \otimes_B$ are both isomorphic to the identity-functors on ${}_A\mathcal{C}$ resp. ${}_B\mathcal{C}$, hence they are inverse equivalences. Furthermore $P \otimes_B$ and $Q \otimes_A$ are $\mathcal{C}$ -functors by Theorem 4.2.

5.4. Theorem. Let $P \in {}_A\mathcal{C}$ be faithfully projective. Then ${}_A[P, -]: {}_A\mathcal{C} \rightarrow {}_A[P, P]\mathcal{C}$ exists and is a $\mathcal{C}$ -equivalence.

PROOF. By definition ${}_A[P, A]$ and ${}_A[P, P] = B$ exist. Furthermore $P \in {}_A\mathcal{C}_B$, $Q := {}_A[P, A] \in {}_B\mathcal{C}_A$ and the hypotheses of Theorem 5.3. are satisfied by the very definition of $Q \otimes_A P \rightarrow B$ and $P \otimes_B Q \rightarrow A$. So $Q \otimes_A -: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ is a $\mathcal{C}$ -equivalence. By Theorem 5.1. we get $Q \otimes_A \cong {}_A[P, -]$.

Let us now apply our theorems to the case where the tensor-product in $\mathcal{C}$ is the (direct) product (example c) of §1). Furthermore assume that each canonical epimorphism $M \times N \rightarrow M \times_A N$ induces a surjective map $M \times N(I) \rightarrow M \times_A N(I)$. This is for example the case if I is projective in the category $\mathcal{C}$. We say that $M \times N \rightarrow M \times_A N$ is *rationally surjective*. Assume that ${}_A\mathcal{C}$ and ${}_B\mathcal{C}$ are $\mathcal{C}$ -equivalent by $P \otimes_B -: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$ and $Q \otimes_A -: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$. Then we have surjective maps $f: P(I) \times Q(I) \cong P \times Q(I) \rightarrow A(I)$ and $g: Q(I) \times P(I) \cong Q \times P(I) \rightarrow B(I)$ such that $(pq)p' = p(qp')$ and $(qp)q' = q(pq')$ if $f(p, q) = pq$ and $g(q, p) = qp$. Let $p_i \in P(I)$, $q_i \in Q(I)$, $i=0, 1$ be chosen such that $p_0 q_0 = 1 \in A(I)$, $q_1 p_1 = 1 \in B(I)$.

Let us now assume that each element in $A(I)$ which has a left inverse has also a right inverse. We wish to show $A \cong B$ as monoids. First we show $p_1 q_1 = 1 \in A$. By definition we have $(p_0 q_1)(p_1 q_0) = p_0(q_1 p_1)q_0 = p_0 q_0 = 1 \in A(I)$, hence $(p_1 q_0)(p_0 q_1) = 1$. Furthermore we have $p_0 q_1 p_1 = p_0$. This implies $p_1 q_1 = 1 \cdot p_1 q_1 = (p_1 q_0 p_0 q_1) p_1 q_1 = p_1 q_0 (p_0 q_1 p_1) q_1 = p_1 q_0 p_0 q_1 = 1 \in A(I)$. Now define morphisms $P(X) \ni p \mapsto pq_1 \in A(X)$ and $A(X) \ni a \mapsto ap_1 \in P$. They are obviously mutually inverse morphisms in ${}_A\mathcal{C}$. Hence $B \cong {}_A[P, P] \cong {}_A[A, A] \cong A$.

As a special case we get

5.5. Corollary: *In the category of sets $\mathcal{S}$ with the product as monoidal category let A be a group or a commutative monoid or finite. Then ${}_A\mathcal{S}$ and ${}_B\mathcal{S}$ are equivalent iff $A \cong B$.*

PROOF. In $\mathcal{S}$ the morphism-sets form an inner hom-functor, so by Theorem 4.3. each equivalence ${}_A\mathcal{S} \cong {}_B\mathcal{S}$ is an $\mathcal{S}$ -equivalence. Furthermore $\{0\}$ is projective in $\mathcal{S}$. If A is a group or commutative or finite then each element which has a left inverse in $A(I)$ has also a right inverse. So all conditions of the previous discussion are satisfied. Hence $A \cong B$. The converse is trivial.

The central part of the Morita Theorems

For this section we will always assume that $\mathcal{C}$ is a symmetric monoidal category. Let us consider $\mathcal{C}$ -functors $\mathcal{U}, \mathcal{V}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ such that there are $P, Q \in {}_B\mathcal{C}_A$ with $\mathcal{C}$ -isomorphisms $\mathcal{U} \cong P \otimes_A$, $\mathcal{V} \cong Q \otimes_A$.

Define a new $\mathcal{C}$ -functor $\mathcal{U} \otimes Y$ for $Y \in \mathcal{C}$ by $\mathcal{U} \otimes Y(M) := \mathcal{U}(M \otimes Y) \cong \cong P \otimes_A (M \otimes Y)$. Because of the symmetry of $\mathcal{C}$ we have $\mathcal{U} \otimes Y(M) \cong (P \otimes Y) \otimes_A M$ hence $\mathcal{U} \otimes Y$ indeed is a $\mathcal{C}$ -functor.

Define $[\mathcal{U}, \mathcal{V}](Y) := \mathcal{C}\text{-Mor}(\mathcal{U} \otimes Y, \mathcal{V})$ as the set of $\mathcal{C}$ -morphisms from $\mathcal{U} \otimes Y$ to $\mathcal{V}$. For $h: Z \rightarrow Y$ in $\mathcal{C}$ define $[\mathcal{U}, \mathcal{V}](h): [\mathcal{U}, \mathcal{V}](Y) \rightarrow [\mathcal{U}, \mathcal{V}](Z)$ by $[\mathcal{U}, \mathcal{V}](h)(\varphi)(M) := (\mathcal{U}(M \otimes Z) \xrightarrow{\mathcal{U}(M \otimes h)} \mathcal{U}(M \otimes Y) \xrightarrow{\varphi(M)} \mathcal{V}(M))$. Then $[\mathcal{U}, \mathcal{V}]$ is a contravariant functor from $\mathcal{C}$ to the category of sets.

6.1. Theorem. *There is a natural isomorphism of functors from $\mathcal{C}$ to the category of sets:*

$$[\mathcal{U}, \mathcal{V}](Y) \cong {}_B\mathcal{C}_A(P \otimes Y, Q).$$

If ${}_B[P, Q]_A$ exists then $[\mathcal{U}, \mathcal{V}](Y) \cong {}_B[P, Q]_A(Y)$.

PROOF. Let $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$. Then define $\varphi \in [\mathcal{U}, \mathcal{V}](Y)$ by

$$\varphi(M) := (\mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{f \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)).$$

For $g \in {}_A\mathcal{C}(M, N)$ we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M & \xrightarrow{f \otimes_A M} & Q \otimes_A M & \cong & \mathcal{V}(M) \\ \downarrow \mathcal{U}(g \otimes Y) & \downarrow (P \otimes Y) \otimes_A g & \downarrow Q \otimes_A g & & \downarrow \mathcal{V}(g) \\ \mathcal{U}(N \otimes Y) \cong (P \otimes Y) \otimes_A N & \xrightarrow{f \otimes_A N} & Q \otimes_A N & \cong & \mathcal{V}(N) \end{array}$$

hence φ is a natural transformation from $\mathcal{U} \otimes Y$ to $\mathcal{V}$. Furthermore the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{U}((M \otimes X) \otimes Y) \cong (P \otimes Y) \otimes_A (M \otimes X) & \xrightarrow{f \otimes_A (M \otimes X)} & Q \otimes_A (M \otimes X) & \cong & \mathcal{V}(M \otimes X) \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathcal{U}(M \otimes Y) \otimes X \cong ((P \otimes Y) \otimes_A M) \otimes X & \xrightarrow{(f \otimes_A M) \otimes X} & (Q \otimes_A M) \otimes X & \cong & \mathcal{V}(M) \otimes X \end{array}$$

Hence φ is a $\mathcal{C}$ -morphism. This defines a map

$$\Sigma : {}_B\mathcal{C}_A(P \otimes Y, Q) \rightarrow [\mathcal{U}, \mathcal{V}](Y).$$

Conversely let $\varphi \in [\mathcal{U}, \mathcal{V}](Y)$ and define $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$ by $f := (P \otimes Y \cong P \otimes_A A \otimes Y \cong \mathcal{U}(A \otimes Y) \xrightarrow{\varphi(A)} \mathcal{V}(A) \cong Q \otimes_A A \cong Q)$. Clearly $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$. To show that $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$ consider the following commutative diagram

$$\begin{array}{ccccccc}
 P \otimes_A (A \otimes Y) \otimes A & \cong & ((P \otimes Y) \otimes_A A) \otimes A & \cong & \mathcal{U}(A \otimes Y) \otimes A & \xrightarrow{\varphi(A) \otimes A} & \mathcal{V}(A) \otimes A \cong (Q \otimes_A A) \otimes A \\
 \parallel & & \parallel & & \parallel & & \parallel & \parallel & \parallel \\
 (P \otimes Y) \otimes A & \cong & (P \otimes Y) \otimes_A (A \otimes A) & \cong & \mathcal{U}((A \otimes A) \otimes Y) & \xrightarrow{\varphi(A \otimes A)} & \mathcal{V}(A \otimes A) \cong Q \otimes_A (A \otimes A) \cong Q \otimes A \\
 \downarrow \gamma_{P \otimes Y} & & \downarrow (P \otimes Y) \otimes_A \mu_A & & \downarrow \mathcal{U}(\mu_A \otimes Y) & & \downarrow \mathcal{V}(\mu_A) & \downarrow Q \otimes_A \mu_A & \downarrow \gamma_Q \\
 P \otimes Y & \cong & (P \otimes Y) \otimes_A A & \cong & \mathcal{U}(A \otimes Y) & \xrightarrow{\varphi(A)} & \mathcal{V}(A) \cong Q \otimes_A A \cong Q
 \end{array}$$

where the morphism from $(P \otimes Y) \otimes A$ to $Q \otimes A$ along the upper side of the diagram is just $f \otimes A$ and the morphism from $P \otimes Y$ to Q along the lower side is f . Hence f is a right A -morphism. So we have a map

$$\Pi : [\mathcal{U}, \mathcal{V}](Y) \rightarrow {}_B\mathcal{C}_A(P \otimes Y, Q).$$

Now

$$\begin{aligned}
 \Pi \Sigma(f) &= (P \otimes Y \cong \mathcal{U}(A \otimes Y) \xrightarrow{\Sigma(f)(A)} \mathcal{V}(A) \cong Q) = \\
 &= (P \otimes Y \cong (P \otimes Y) \otimes_A A \xrightarrow{f \otimes_A A} Q \otimes_A A \cong Q) = f
 \end{aligned}$$

and

$$\begin{aligned}
 \Sigma \Pi(\varphi)(M) &= (\mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{\pi(\varphi) \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\
 &= (\mathcal{U}(M \otimes Y) \cong \mathcal{U}(A \otimes Y) \otimes_A M \xrightarrow{\varphi(A) \otimes_A M} \mathcal{V}(A) \otimes_A M \cong \mathcal{V}(M)) = \varphi(M)
 \end{aligned}$$

since φ is a $\mathcal{C}$ -morphism. Hence we have $[\mathcal{U}, \mathcal{V}](Y) \cong {}_B\mathcal{C}_A(P \otimes Y, Q)$.

It remains to show that this isomorphism is a natural transformation. Let $h: Z \rightarrow Y$ be in $\mathcal{C}$. Then

$$\begin{array}{ccc}
 {}_B\mathcal{C}_A(P \otimes Y, Q) & \xrightarrow{\Sigma(Y)} & [\mathcal{U}, \mathcal{V}](Y) \\
 \downarrow {}_B\mathcal{C}_A(P \otimes h, Q) & & \downarrow [\mathcal{U}, \mathcal{V}](h) \\
 {}_B\mathcal{C}_A(P \otimes Z, Q) & \xrightarrow{\Sigma(Z)} & [\mathcal{U}, \mathcal{V}](Z)
 \end{array}$$

commutes since for $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$ we have

$$\begin{array}{ccccc}
 \mathcal{U}(M \otimes Z) \cong (P \otimes Z) \otimes_A M & & & & \\
 \downarrow \mathcal{U}(M \otimes h) & \downarrow (P \otimes h) \otimes_A M & \searrow (f(P \otimes h)) \otimes_A M & & \\
 \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M & \xrightarrow{f \otimes_A M} & Q \otimes_A M \cong \mathcal{V}(M) & &
 \end{array}$$

commutative and thus

$$\begin{aligned}
 & ([\mathcal{U}, \mathcal{V}](h) \circ \Sigma(Y))(f)(M) = \\
 & = (\mathcal{U}(M \otimes Z) \xrightarrow{\mathcal{U}(M \otimes h)} \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{f \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\
 & = (\mathcal{U}(M \otimes Z) \cong (P \otimes Z) \otimes_A M \xrightarrow{(f(P \otimes h)) \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\
 & = (\Sigma(Z) \circ {}_B\mathcal{C}_A(P \otimes h, Q))(f)(M).
 \end{aligned}$$

For this theorem we have two applications. Before we discuss them, we have to introduce the notion of the center of a monoid.

It is clear that ${}_A\mathcal{C}(A, A) \cong A(I)$ as monoids in the category of sets. The isomorphism is given by

$${}_A\mathcal{C}(A, A) \ni f \mapsto f(1) = f\eta \in A(I)$$

$$A(I) \ni a \mapsto (A(X) \ni b \mapsto ba \in A(X)) \in {}_A\mathcal{C}(A, A).$$

Now those elements $a \in A(I)$ which commute with all $b \in A(X)$ for all X induce in ${}_A\mathcal{C}(A, A)$ precisely the $A-A$ -morphisms ${}_A\mathcal{C}_A(A, A)$, which then is a commutative monoid. So a possible definition of the center of A could be ${}_A\mathcal{C}_A(A, A)$. But this is only a set, not an object in $\mathcal{C}$. A possible generalization to an object in $\mathcal{C}$ is ${}_A[A, A]_A$ if this exists. If it does not exist we know at least the functor represented by this object. So we define the *center of A* as a functor from $\mathcal{C}$ to $\mathcal{S}$, the category of sets, by $\text{Cent}(A)(X) := {}_A\mathcal{C}_A(A \otimes X, A)$. If ${}_A[A, A]_A$ exists we have $\text{Cent}(A)(X) \cong {}_A[A, A]_A(X)$.

As in § 3 $\text{Cent}(A)$ can only be defined in a symmetric monoidal category in contrast to ${}_A\mathcal{C}_A(A, A)$. In § 2 we showed that ${}_I\mathcal{C} \cong \mathcal{C}$ and $\mathcal{C} \cong \mathcal{C}_I$ hence $\mathcal{C}(I, I) \cong {}_I\mathcal{C}_I(I, I)$ is a commutative monoid [18, Theorem 1] for (possibly nonsymmetric) monoidal categories.

Let A be a monoid in a symmetric monoidal category $\mathcal{C}$. Let ${}_A\text{Id}: {}_A\mathcal{C} \rightarrow {}_A\mathcal{C}$ denote the identity functor. Then ${}_A\text{Id}$ is clearly a $\mathcal{C}$ -functor and ${}_A\text{Id} \cong A \otimes_A$ as $\mathcal{C}$ -functors.

6.2. Theorem. $\text{Cent}(A) \cong [{}_A Id, {}_A Id]$.

PROOF. $\text{Cent}(A)(X) = {}_A \mathcal{C}_A(A \otimes X, A) \cong [{}_A Id, {}_A Id](X)$.

The isomorphism of Theorem 6.2. is only an isomorphism of objects in $\mathcal{C}$. But there is an additional structure, a multiplication on these functors. If they were representable the representing objects in $\mathcal{C}$ would be monoids. The multiplicative structure on ${}_A[A, A]_A$ as it has been studied in § 3 is reflected in ${}_A \mathcal{C}_A(A \otimes X, A)$ by the commutative diagram

$$\begin{array}{ccc} {}_A[A, A]_A(X) \times {}_A[A, A]_A(Y) & \rightarrow & {}_A[A, A]_A(X \otimes Y) \\ \parallel & & \parallel \\ {}_A \mathcal{C}_A(A \otimes X, A) \times {}_A \mathcal{C}_A(A \otimes Y, A) & \rightarrow & {}_A \mathcal{C}_A(A \otimes X \otimes Y, A) \end{array}$$

where the lower map is given by

$$(f, g) \mapsto (A \otimes X \otimes Y \xrightarrow{f \otimes Y} A \otimes Y \xrightarrow{g} A).$$

The unit is described by

$$I(X) \ni f \mapsto (A \otimes X \xrightarrow{A \otimes f} A \otimes I \cong A) \in {}_A \mathcal{C}_A(A \otimes X, A).$$

$[{}_A Id, {}_A Id]$ carries a multiplicative structure via

$$[{}_A Id, {}_A Id](X) \times [{}_A Id, {}_A Id](Y) \xrightarrow{T} [{}_A Id, {}_A Id](X \otimes Y)$$

by
$$T(\varphi, \psi) = ({}_A Id \otimes X \otimes Y \xrightarrow{\varphi \otimes Y} {}_A Id \otimes Y \xrightarrow{\psi} {}_A Id)$$

and there is a unit

$$I(X) \ni f \mapsto ({}_A Id \otimes X \xrightarrow{{}_A Id \otimes f} {}_A Id \otimes I \cong {}_A Id) \in [{}_A Id, {}_A Id](X).$$

Using the isomorphism of Theorem 6.1. it is easy to see that they are compatible with the multiplication and the unit map. Hence the isomorphism of Theorem 6.2. is a „monoid isomorphism”.

6.3. Corollary: Let ${}_A \mathcal{C}$ and ${}_B \mathcal{C}$ be $\mathcal{C}$ -equivalent. Then $\text{Cent}(A) \cong \text{Cent}(B)$ as functors from $\mathcal{C}$ to $\mathcal{S}$. If both functors are representable then the two representing objects are isomorphic as commutative monoids in $\mathcal{C}$:

$${}_A[A, A]_A \cong {}_B[B, B]_B.$$

PROOF. We show $[{}_A Id, {}_A Id] \cong [{}_B Id, {}_B Id]$. Let $\mathcal{F}: {}_A \mathcal{C} \rightarrow {}_B \mathcal{C}$ the given $\mathcal{C}$ -equivalence. First we show

$$[{}_A Id, {}_A Id](X) \cong [\mathcal{F}, \mathcal{F}](Y),$$

or

$$\mathcal{C}\text{-Mor}({}_A Id \otimes Y, {}_A Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

Let $\varphi: {}_A Id \otimes Y \rightarrow {}_A Id$ be a natural transformation. Define $\mathcal{F} \circ \varphi: \mathcal{F} \otimes Y \rightarrow \mathcal{F}$ by $\mathcal{F} \circ \varphi(M): \mathcal{F}(M \otimes Y) \rightarrow \mathcal{F}(M)$ as $\mathcal{F} \circ \varphi(M) = \mathcal{F}(\varphi(M))$. Since $\mathcal{F}$ is an equivalence it is clear that $\varphi \mapsto \mathcal{F} \circ \varphi$ is a bijection between the sets of natural transformations. Now we show that φ is a $\mathcal{C}$ -morphism iff $\mathcal{F} \circ \varphi$ is a $\mathcal{C}$ -morphism. φ is a $\mathcal{C}$ -morphism iff the diagrams

$$\begin{array}{ccc}
M \otimes X \otimes Y & \xrightarrow{\varphi(M \otimes X)} & M \otimes X \\
\parallel & & \parallel \\
M \otimes Y \otimes X & \xrightarrow{\varphi(M) \otimes X} & M \otimes X
\end{array}$$

commute. $\mathcal{F} \circ \varphi$ is a $\mathcal{C}$ -morphism iff the outer diagrams of

$$\begin{array}{ccc}
\mathcal{F}(M \otimes X \otimes Y) & \xrightarrow{\mathcal{F} \circ \varphi(M \otimes X)} & \mathcal{F}(M \otimes X) \\
\parallel & & \parallel \\
\mathcal{F}((M \otimes Y) \otimes X) & \xrightarrow{\mathcal{F}(\varphi(M) \otimes X)} & \mathcal{F}(M \otimes X) \\
\parallel & & \parallel \\
\mathcal{F}(M \otimes Y) \otimes X & \xrightarrow{\mathcal{F} \circ \varphi(M) \otimes X} & \mathcal{F}(M) \otimes X
\end{array}$$

commute where the lower part commutes in any case since $\mathcal{F}$ is a $\mathcal{C}$ -functor. But the upper part commutes iff the previous diagram commutes. Hence

$$\mathcal{C}\text{-Mor}({}_A Id \otimes Y, {}_A Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

Now we show $[{}_B Id, {}_B Id](Y) \cong [\mathcal{F}, \mathcal{F}](Y)$ or

$$\mathcal{C}\text{-Mor}({}_B Id \otimes Y, {}_B Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

It is clear that the correspondence between $\varphi: {}_B Id \otimes Y \rightarrow {}_B Id$ and $\varphi \mathcal{F} \circ: \mathcal{F} \otimes Y \rightarrow \mathcal{F}$ with

$$(\varphi \circ \mathcal{F})(M) := (\mathcal{F}(M \otimes Y) \cong \mathcal{F}(M) \otimes Y \xrightarrow{\varphi(\mathcal{F}(M))} \mathcal{F}(M))$$

induces an isomorphism between the sets of natural transformation, since $\mathcal{F}$ is an equivalence. Furthermore φ is a $\mathcal{C}$ -morphism iff the diagrams

$$\begin{array}{ccc}
N \otimes X \otimes Y & \xrightarrow{\varphi(N \otimes X)} & N \otimes X \\
\parallel & & \parallel \\
N \otimes Y \otimes X & \xrightarrow{\varphi(N) \otimes X} & N \otimes X
\end{array}$$

commute. On the other hand $\varphi \circ \mathcal{F}$ is a $\mathcal{C}$ -morphism iff the outer diagrams

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes X \otimes Y) & \xrightarrow{\varphi \circ \mathcal{F}(M \otimes X)} & \mathcal{F}(M \otimes X) \\
 \parallel & & \parallel \\
 \mathcal{F}(M \otimes X) \otimes Y & \xrightarrow{\varphi(\mathcal{F}(M \otimes X))} & \mathcal{F}(M \otimes X) \\
 \parallel & & \parallel \\
 \mathcal{F}(M) \otimes Y \otimes X & \xrightarrow{\varphi(\mathcal{F}(M) \otimes X)} & \mathcal{F}(M) \otimes X \\
 \parallel & & \parallel \\
 \mathcal{F}(M \otimes Y) \otimes X & \xrightarrow{\varphi \circ \mathcal{F}(M) \otimes X} & \mathcal{F}(M) \otimes X
 \end{array}$$

commute. The first and third part commute by definition. In the middle part take into account that $\mathcal{F}$ is a $\mathcal{C}$ -functor. Then it commutes iff the previous diagram for φ commutes. Hence $[_B Id, _B Id](Y) \cong [\mathcal{F}, \mathcal{F}](Y) \cong [_A Id, _A Id](Y)$.

The reader can easily verify that these isomorphisms are natural isomorphisms in Y . Furthermore they preserve the „multiplication“ given by composition of morphisms just before Corollary 6.3. They also preserve the „unit“. Hence

$$_A[A, A]_A \cong _B[B, B]_B$$

as monoids (if they exist) or

$$\text{Cent}(A) \cong \text{Cent}(B)$$

with the multiplicative structure.

6.4. Corollary: Let $\mathcal{U}: {}_A\mathcal{C} \rightarrow \mathcal{C}$ be the underlying functor. Then $[\mathcal{U}, \mathcal{U}](Y) \cong \cong A^{\text{op}}(Y)$ natural in $Y \in \mathcal{C}$ and compatible with the multiplication on both sides.

PROOF. By Theorem 6.1. and the fact $\mathcal{U} \cong A \otimes_A$ we have $[\mathcal{U}, \mathcal{U}](Y) \cong \cong \mathcal{C}_A(A \otimes Y, A) \cong A^{\text{op}}(Y)$ as left multiplications and these isomorphisms are natural in Y and compatible with the multiplication.

So we have seen that just from the knowledge of the underlying functor $\mathcal{U}$ we may regain the monoid A up to an isomorphism.

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