

Some extended Hermite—Fejér interpolation processes and their convergence

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1. P. Szász [3] introduced an interpolation formula what he calls extended Hermite—Fejér interpolation. According to him for given $n+m$ distinct points

$$(1.1) \quad x_1, x_2, \dots, x_n; \quad \xi_1, \xi_2, \dots, \xi_m,$$

there exists a unique polynomial $S(x)$ of degree $\leq 2n+m-1$ satisfying the following conditions:

$$(1.2) \quad \begin{aligned} S(x_k) &= f(x_k), \quad S'(x_k) = 0, \quad k = 1, 2, \dots, n, \\ S(\xi_j) &= f(\xi_j), \quad j = 1, 2, \dots, m. \end{aligned}$$

Szász himself constructed interpolatory polynomials when $m=1$, $\xi_1=1$ and x_k 's are the zeros of $P_n(x)$, n th Legendre polynomial or $P_n^{(1/2, -1/2)}(x)$, n th Jacobi polynomial and proved the convergence for any continuous function $f(x)$.

Further SAXENA [4] has investigated the convergence in the case, when $m=1$, $\xi_1=-1$ and x_k 's are the zeros of $P_n^{(-1/2, 1/2)}(x)$, n th Jacobi polynomial.

In 1965 MERLI [2] constructed polynomials when $m=1$, $\xi_1=0$ and x_k 's are zeros of $T_n(x)$, n th Tchebycheff polynomial of the first kind, n even, and proved the convergence to $f(x)$, which is given by

$$f(x) = c + x^2 \Phi(x)$$

where C is any constant and $\Phi(x)$ is continuous function in $[-1, 1]$. Later FONTANELLA [1] proved the Merli's theorem for the function $f(x)$, where $f(x) = c + |x|^\alpha \Phi(x)$, $\alpha > 0$.

The case when $m=2$, $\xi_1=1$ & $\xi_2=-1$ has also been studied by Szász [5]. Now it is interesting to consider the following cases:

- i) When $m=2$, $\xi_1=1$ and $\xi_2=0$;
 - ii) When $m=2$, $\xi_1=-1$ and $\xi_2=0$;
- and
- iii) When $m=3$, $\xi_1=1$, $\xi_2=-1$ and $\xi_3=0$.

2. Case i) When $m=2$, $\xi_1=1$ and $\xi_2=0$.

The interpolation polynomials $R_n(f, x)$ of degree $\leq 2n+1$ satisfying the properties

$$(2.1) \quad \begin{cases} R_n(f, 1) = f(1) & R_n(f, 0) = 0 \\ R_n(f, x_k) = f(x_k) & R'_n(f, x_k) = 0, \quad k = 1, 2, \dots, n \end{cases}$$

is given by

$$(2.2) \quad R_n(f, x) = f(1) \frac{W_n^2(x)}{W_n^2(1)} x + \sum_{k=1}^n f(x_k) \frac{x(1-x)}{x_k(1-x_k)} \left[1 - \left\{ \frac{1-2x_k}{x_k(1-x_k)} + \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x),$$

where

$$(2.3) \quad \begin{cases} W_n(x) = \prod_{k=1}^n (x-x_k), \\ l_k(x) = \frac{W_n(x)}{W_n'(x_k)(x-x_k)}, \quad k = 1, 2, \dots, n. \end{cases}$$

Theorem 1. Let $f(x) = c + |x|^\alpha \Phi(x)$, where c is any constant, $\alpha \geq 2$, $\Phi(x)$ is continuous function in $[-1, 1]$ and x_k 's are the zeros of $P_n^{(1/2, -1/2)}(x)$, n th Jacobi polynomial. Then the sequence of polynomials $\{R_n(f, x)\}$ given by (2.2), converges uniformly to $f(x)$ in $[-1, 1]$.

The polynomial $P_n^{(1/2, -1/2)}(x)$ is identical with

$$(2.4) \quad W_n(x) = \text{const.} \left[\frac{\sin(2n+1)\Theta/2}{\sin \Theta/2} \right]_{x=\cos \Theta},$$

having the zeros

$$x_k = \cos \frac{2k\pi}{2n+1}, \quad k = 1, 2, \dots, n.$$

This polynomial $W_n(x)$ satisfies the differential equation

$$(1-x^2)W_n''(x) - (1+2x)W_n'(x) + n(n+1)W_n(x) = 0,$$

which reduces (2.2) as

$$R_n(f, x) = f(1) \frac{W_n^2(x)}{(2n+1)^2} x + \sum_{k=1}^n f(x_k) \frac{x(1-x)(2x_k - x_k^3 - x)}{x_k^2(1-x_k^2)(1-x_k)} l_k^2(x).$$

In particular for¹⁾ $f(x) = |x|^\alpha \Phi(x)$, $\alpha \geq 2$, we have

$$(2.5) \quad R_n(f, x) = \Phi(1) \frac{W_n^2(x)}{(2n+1)^2} x + \sum_{k=1}^n \Phi(x_k) |x_k|^{\alpha-2} \frac{x(1-x)(2x_k - x_k^3 - x)}{(1+x_k)(1-x_k)} l_k^2(x).$$

Now a quasi Hermite—Fejér interpolation process of degree $\leq 2n$, satisfying the conditions:

$$R_n^*(f, 1) = f(1), \quad R_n^*(f, x_k) = f(x_k), \quad R_n^{*'}(f, x_k) = 0 \quad k = 1, 2, \dots, n,$$

is given by

$$(2.6) \quad R_n^*(f, x) = f(1) \frac{W_n^2(x)}{W_n^2(1)} + \sum_{k=1}^n f(x_k) \frac{1-x}{1-x_k} \left[1 + \left\{ \frac{1}{1-x_k} - \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x).$$

¹⁾ Without no loss of generality, now we will always take $C=0$.

Using (2.4) and $f(x) = |x|^\alpha \Phi(x)$, $\alpha \geq 2$ we have at once

$$(2.7) \quad R_n^*(f, x) = \Phi(1) \frac{W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^n \Phi(x_k) \frac{1-x}{1-x_k} |x_k|^\alpha \left(\frac{1-xx_k}{1-x_k^2} \right) l_k^2(x).$$

Subtracting (2.5) from (2.7), we obtain

$$R_n^*(f, x) - R_n(f, x) = \Phi(1) \frac{(1-x)W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^n \Phi(x_k) \frac{|x_k|^{\alpha-2}(1-x)}{(1-x_k)^2(1+x_k)W_n^2(x_k)}.$$

Since $0 \leq (1-x)W_n^2(x) \leq 2^2$ and $|\Phi(x)| \leq M$ in $[-1, 1]$, it follows that for $\alpha \geq 2$

$$|R_n^*(f, x) - R_n(f, x)| < \frac{2M}{(2n+1)^2} + 2M \sum_{k=1}^n \frac{1}{(1-x_k^2)(1+x_k)W_n^2(x_k)}.$$

But³⁾

$$\sum_{k=1}^n \frac{1}{(1-x_k^2)(1+x_k)W_n^2(x_k)} = \frac{2n}{(2n+1)^2}$$

therefore,

$$(2.8) \quad \lim_{n \rightarrow \infty} R_n(f, x) = \lim_{n \rightarrow \infty} R_n^*(f, x).$$

Further using the well known result of Szász [3] we have

$$(2.9) \quad \lim_{n \rightarrow \infty} R_n^*(f, x) = f(x), \quad -1 \leq x \leq 1$$

for $f(x) = |x|^\alpha \Phi(x)$, continuous in $[-1, 1]$ ⁴⁾. Combining (2.8) and (2.9) we have the theorem 1.

Theorem 2. Let $f(x) = c + |x|^\alpha \Phi(x)$ where C is any constant and $\Phi(x)$ continuous in $[-1, 1]$. Then for the sequence of polynomials (2.2) constructed on the zeros of $P_n(x)$, n th Legendre polynomial, the relation

$$R_n(f, x) \rightarrow f(x) \quad \text{for} \quad -1 < x \leq 1$$

holds.

The proof of Theorem 2 runs exactly on the same lines as the proof of theorem 1. So we omit the details.

3. Case ii) when $m=2$, $\xi_1=-1$ and $\xi_2=0$.

The interpolation polynomial $H_n(f, x)$ of degree $\leq 2n+1$ with the properties:

$$\begin{aligned} H_n(f, -1) &= f(-1) & H_n(f, 0) &= 0 \\ H_n'(f, x_k) &= f(x_k) & H_n''(f, x_k) &= 0, \quad k = 1, 2, \dots, n \end{aligned}$$

is given by

$$(3.1) \quad \begin{aligned} H_n(f, x) &= \\ &= f(-1) \frac{W_n^2(x)}{W_n^2(-1)} x + \sum_{k=1}^n f(x_k) \frac{x(1+x)}{x_k(1+x_k)} \left[1 - \left\{ \frac{1+2x_k}{x_k(1+x_k)} + \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x) \end{aligned}$$

where $W_n(x)$ and $l_k(x)$ are given by (2.3).

³⁾ See Szász [3]

³⁾ See Szász [3]

⁴⁾ See for example holds G. VITAL, G. SAUSONE, *Moderna teoria delle funzioni di variabile reale*. Bologna (1952).

Theorem 3. If $\Phi(x)$ is continuous in $[-1, 1]$ and $f(x) = C + |x|^\alpha \Phi(x)$, $\alpha \geq 2$ and x_k 's denote the zeros of Jacobi polynomial $P_n^{(-1/2, 1/2)}(x)$, then the sequence of polynomials (3.1) converges uniformly to $f(x)$ in $[-1, 1]$.

The polynomial $P_n^{(-1/2, 1/2)}(x)$ is identical with

$$(3.2) \quad W_n(x) = \text{const} \frac{\cos \frac{2n+1}{2} \Theta}{\cos \Theta/2}, \quad x = \cos \Theta$$

having the zeros

$$x_k = \cos \frac{2k-1}{2n+1} \Pi, \quad k = 1, 2, \dots, n.$$

The differential equation satisfied by $W_n(x)$ in (3.2) is

$$(1-x^2)W_n''(x) + (1-2x)W_n'(x) + n(n+1)W_n(x) = 0$$

which reduces (3.1) for $f(x) = |x|^\alpha \Phi(x)$ as

$$(3.3) \quad H_n(f, x) = \Phi(-1) \frac{W_n^2(x)}{-(2n+1)^2} x + \sum_{k=1}^n \Phi(x_k) \frac{x(1+x)x_k|x_k|^{\alpha-2}}{(1+x_k)^2(1-x_k)} (2x_k - x_k^3 - x) l_k^2(x).$$

Now a quasi Hermite—Fejér polynomial satisfying the condition:

$$H_n^*(f, -1) = f(-1), \quad H_n^*(f, x_k) = 0, \quad H_n^{*'}(f, x_k) = 0, \quad k = 1, 2, \dots, n,$$

is given by

$$H_n^*(f, x) = f(-1) \frac{W_n^2(x)}{W_n^2(-1)} + \sum_{k=1}^n f(x_k) \frac{1+x}{1+x_k} \left[1 - \left\{ \frac{1}{1+x_k} + \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x).$$

For $W_n(x)$ given by (3.2) and $f(x) = |x|^\alpha \Phi(x)$, $\alpha \geq 2$ we have

$$(3.4) \quad H_n^*(f, x) = \Phi(-1) \frac{W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^n \Phi(x_k) |x_k|^\alpha \frac{(1+x)(1-xx_k)}{(1-x_k)^2(1+x_k)} l_k^2(x).$$

SAXENA [4] proved that

$$(3.5) \quad H_n^*(f, x) \rightarrow f(x), \quad -1 \leq x \leq 1$$

as $n \rightarrow \infty$ when $f(x)$ is continuous in $[-1, 1]$.

Equation (3.3) and (3.4) yield

$$H_n^*(f, x) - H_n(f, x) = \Phi(-1) \frac{(1+x)W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^n \Phi(x_k) |x_k|^{\alpha-2} \frac{(1+x)}{(1+x_k)(1-x_k^2)W_n'^2(x_k)}.$$

Since $|\Phi(x)| \leq M$ for $-1 \leq x \leq 1$, $\alpha \geq 2$ and⁵⁾

$$0 < (1+x)W_n^2(x) \leq 2, \quad \sum_{k=1}^n \frac{1}{(1-x_k^2)(1+x_k)W_n'^2(x_k)} = \frac{2n}{(2n+1)^2}.$$

⁵⁾ See SAXENA [4].

Hence

$$(3.6) \quad H_n(f, x) \rightarrow H_n^*(f, x) \quad \text{as } n \rightarrow \infty.$$

Equation (3.5) and (3.6) together completes the proof of theorem 3.

4. Case iii), when $m=3$, $\xi_1=1$, $\xi_2=-1$ and $\xi_3=0$.

The interpolation polynomial $Q_n(f, x)$ of degree $\leq 2n+2$ with the properties:

$$(4.1) \quad \begin{aligned} Q_n(f, 1) &= f(1), & Q_n(f, 0) &= 0, & Q_n(f, -1) &= f(-1) \\ Q_n(f, x_k) &= f(x_k), & Q'_n(f, x_k) &= 0, & k &= 1, 2, \dots, n, \end{aligned}$$

is given by

$$(4.2) \quad \begin{aligned} Q_n(f, x) &= f(1) \frac{W_n^2(x)}{W_n^2(1)} \cdot \frac{(1+x)x}{2} + f(-1) \frac{W_n^2(x)}{W_n^2(-1)} \cdot \frac{(x-1)x}{2} + \\ &+ \sum_{k=1}^n f(x_k) \frac{x(1-x^2)}{x_k(1-x_k^2)} \left[1 - \left\{ \frac{1}{x_k} - \frac{2x_k}{1-x_k^2} + \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x), \end{aligned}$$

where $l_k(x)$ and $W_n(x)$ are given by (2.3).

Theorem 4. Let $f(x) = C + |x|^\alpha \Phi(x)$, where C is any constant, $\alpha \geq 2$, $\Phi(x)$ continuous function in $[-1, 1]$ and x_k 's be the zeros of $U_n(x)$, n th Tchebycheff polynomial of second kind, n even. Then the sequence of polynomials (4.2) converges uniformly to $f(x)$ in $[-1, 1]$.

The differential equation satisfied by $W_n(x) = U_n(x)$ is

$$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0$$

from which we at once have

$$\begin{aligned} Q_n(f, x) &= f(1) \frac{U_n^2(x)(1+x)x}{2(n+1)^2} + f(-1) \frac{U_n^2(x)(x-1)x}{2(n+1)^2} + \\ &+ \sum_{k=1}^n f(x_k) \frac{x(1-x^2)(2x_k-x_k^3-x)}{x_k^2(1-x_k^2)^2} l_k^2(x). \end{aligned}$$

Taking $f(x) = |x|^\alpha \Phi(x)$, $\alpha \geq 2$, we have

$$(4.3) \quad \begin{aligned} Q_n(f, x) &= \Phi(1) \frac{U_n^2(x)}{2(n+1)^2} x(1+x) + \Phi(-1) \frac{U_n^2(x)}{2(n+1)^2} x(x-1) + \\ &+ \sum_{k=1}^n \Phi(x_k) |x_k|^{\alpha-2} \frac{x(1-x^2)(2x_k-x_k^3-x)}{(1-x_k^2)} l_k^2(x). \end{aligned}$$

Now the quasi-Hermite—Fejér interpolation polynomial $Q_n^*(f, x)$ satisfying the properties

$$\begin{aligned} Q_n^*(f, 1) &= f(1), & Q_n^*(f, -1) &= f(-1) \\ Q_n^*(f, x_k) &= f(x_k), & Q_n^{*'}(f, x_k) &= 0, & k &= 1, 2, \dots, n, \end{aligned}$$

is given by

$$\begin{aligned} Q_n^*(f, x) &= f(1) \frac{1+x}{2} \frac{W_n^2(x)}{W_n^2(1)} + f(-1) \frac{1-x}{2} \frac{W_n^2(x)}{W_n^2(-1)} + \\ &+ \sum_{k=1}^n f(x_k) \frac{1-x^2}{1-x_k^2} \left[1 + \left\{ \frac{2x_k}{1-x_k^2} - \frac{W_n''(x_k)}{W_n'(x_k)} \right\} (x-x_k) \right] l_k^2(x). \end{aligned}$$

Which for $W_n(x) = U_n(x) \& f(x) = |x|^\alpha \Phi(x)$, $\alpha \geq 2$ takes the form

$$(4.4) \quad \begin{aligned} Q_n^*(f, x) = & \Phi(1) \frac{1+x}{2} \frac{U_n^2(x)}{(n+1)^2} + \Phi(-1) \frac{1-x}{2} \frac{U_n^2(x)}{(n+1)^2} + \\ & + \sum_{k=1}^n \Phi(x_k) |x_k|^\alpha \frac{(1-x)^2(1-xx_k)}{(1-x_k^2)^2} l_k^2(x). \end{aligned}$$

From (4.3) and (4.4), we get

$$Q_n^*(f, x) - Q_n(f, x) = [\Phi(1) - \Phi(-1)] \frac{1-x^2}{2(n+1)^2} U_n^2(x) + \sum_{k=1}^n \Phi(x_k) |x_k|^{\alpha-2} \frac{(1-x^2)U_n^2(x)}{(n+1)^2}.$$

Further we have

$$|Q_n^*(f, x) - Q_n(f, x)| < |\Phi(1) - \Phi(-1)| \frac{(1-x^2)U_n^2(x)}{2(n+1)^2} + \sum_{k=1}^n |\Phi(x_k)| |x_k|^{\alpha-2} \frac{(1-x^2)U_n^2(x)}{(n+1)^2}.$$

Since $|\Phi(x)| \leq M$, $\alpha \geq 2$, $0 < (1-x^2)U_n^2(x) < 1$ for $-1 \leq x \leq 1$, therefore

$$(4.5) \quad \lim_{n \rightarrow \infty} Q_n(f, x) = \lim_{n \rightarrow \infty} Q_n^*(f, x).$$

But theorem of Szász [5] gives

$$(4.6) \quad Q_n^*(f, x) \rightarrow f(x) \quad \text{as } n \rightarrow \infty.$$

From (4.5) and (4.6) we have the theorem 4.

Theorem 5. If $f(x) = C + |x|^\alpha \Phi(x)$ where C is any constant, $\alpha \geq 2$ and $\Phi(x)$ is continuous in $[-1, 1]$, then the sequence of polynomials (4.2) constructed on the zeros of $P_n(x)$, n th Legendre polynomial converges uniformly to $f(x)$ in $[-1, 1]$.

The proof of this theorem runs exactly on the same lines as the proof of the theorem 5. So we omit the details.

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