Effective estimates for the integer solutions of norm form and discriminant form equations

By K. GYŐRY and Z. Z. PAPP (Debrecen)

Dedicated to the memory of Professor Andor Kertész

1. Introduction

In this paper we generalize Baker's famous theorem on the Thue equation [2] to diophantine equations in an arbitrary number of unknowns. Our main result (Theorem 1) includes Baker and Coates's theorem on the generalized Thue equation [6] (see also BAKER [3], [4]) and implies a general result of the first named author [13] (see also [14]) concerning discriminant form equations. As a consequence of Theorem 1 we make effective, for a wide class of norm forms, a well-known ineffective theorem of Schmidt on norm form equations [25].

$\mathfrak{p}$-adic generalizations and further applications are given in our joint papers [20] and [21].

The proof of our Theorem 1 depends on a recent result on linear diophantine equations in algebraic integers of bounded norm [18] whose proof is based on an explicit estimate of VAN DER POORTEN and LOXTON [23] for linear forms in the logarithms of algebraic numbers. Our explicit bounds given for the sizes of the integer solutions of the discussed equations are, in some respects, the best possible in terms of certain parameters.

2. Representation of algebraic numbers by decomposable forms in several variables

Let $F(x, y)=a_0x^n+a_1x^{n-1}y+\dots+a_ny^n\in\mathbf{Z}[x, y]$ be an irreducible binary form of degree $n\geqq 3$ with $\max_{0\leqq i\leqq n}|a_i|\leqq H$, and let b be a non-zero rational integer. By a celebrated theorem of BAKER [2] all solutions in integers x, y of the Thue equation

$$F(x, y) = b \tag{1}$$

satisfy

$$\max(|x|, |y|) < \exp\{n^{v^2}H^{vn^3}+(\log|b|)^{n+2}\}, \tag{2}$$

where $v=32n(n+2)^2$. (2) was later considerably improved in b and H among others by FELDMAN [10], SPRINDŽUK [29], [30] and STARK [33].

Let now K be an algebraic number field with degree k, let $\alpha_1, \dots, \alpha_n$ be $n\geqq 3$ distinct algebraic integers in K, and let β be any non-zero algebraic integer in K.

As a generalization of the above result BAKER and COATES [6] (see also BAKER [3], [4]) showed that all solutions of the generalized Thue equation

$$(3) \qquad (x-\alpha_1 y)\dots(x-\alpha_n y) = \beta$$

in algebraic integers x, y of K satisfy[1])

$$(4) \qquad \max(\overline{|x|}, \overline{|y|}) < \exp\{(k\mathscr{H})^{(10k)^5}\},$$

where $\mathscr{H}$ denotes the maximum of the heights[2]) of $\alpha_1, \dots, \alpha_n, \beta$ and some algebraic integer generating K. $\mathfrak{p}$-adic generalizations have been obtained by SPRINDŽUK and KOTOV [31], [32] and KOTOV [22].

Our Theorem 1 generalizes the above-quoted theorems of BAKER [2] and BAKER and COATES [6] to diophantine equations in an arbitrary number of unknowns. To state our main result we need the following definition. *A system $\mathscr{L}$ of $n \geqq 2$ linear forms $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$ in $\mathbf{x}=(x_1, \dots, x_m)$ with algebraic coefficients will be called triangularly connected or $\triangle$-connected, if for any distinct i, j with $1 \leqq i, j \leqq n$ there is a sequence $L_i = L_{i_1}, L_{i_2}, \dots, L_{i_s} = L_j$ in $\mathscr{L}$ such that for each t with $1 \leqq t \leqq$ $\leqq s-1$ $L_{i_t}, L_{i_{t+1}}$ have a linear combination with non-zero algebraic coefficients which belongs to $\mathscr{L}$* [3]). In particular when $m=2$, every system $\mathscr{L}$ which contains at least three pairwise nonproportional linear forms is $\triangle$-connected.

Theorem 1. *Let $L \subseteq K$ be algebraic number fields and let $0 \neq \beta \in \mathbf{Z}_K$* [4]) *with $\overline{|\beta|} \leqq B$, $|N_{L(\beta)/Q}(\beta)| \leqq B^*$. Further, let $m, n \geqq 2$ be integers and let α_{ij} $(1 \leqq i \leqq n,$ $1 \leqq j \leqq m)$ be algebraic integers in K with sizes at most A $(\geqq 2)$. Let N $(\geqq 2)$, R and D $(\geqq 2)$ denote upper bounds for the degrees, the regulators and the absolute values of the discriminants of the fields $K_{ijl} = L(\alpha_{i1}, \dots, \alpha_{im}, \alpha_{j1}, \dots, \alpha_{jm}, \alpha_{l1}, \dots, \alpha_{lm})$, $i, j, l=$ $=1, \dots, n$. Suppose that the linear forms $L_i(\mathbf{x}) = \alpha_{i1}x_1 + \dots + \alpha_{im}x_m$ $(1 \leqq i \leqq n)$ form a $\triangle$-connected system, and that the system of equations*

$$(5) \qquad L_i(\mathbf{x}) = 0, \quad i = 1, \dots, n$$

has no solutions $\mathbf{x} \neq 0$ with components in L. Then all solutions $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{Z}_L^m$ of the equation

$$(6) \qquad L_1(\mathbf{x}) \dots L_n(\mathbf{x}) = \beta$$

satisfy

$$(7) \qquad \max_{1 \leqq i \leqq m} \overline{|x_i|} < B^{1/n} \exp\{mn(8N)^{32(N+1)}(R \log R^*)^2(R + \log AB^*)\}$$

and

$$(8) \qquad \max_{1 \leqq i \leqq m} \overline{|x_i|} < B^{1/n} \exp\{mn(8N)^{30(N+1)} D(\log D)^{3N-1}(D^{1/2} + \log AB^*)\},$$

where $R^* = \max(R, e)$.

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[1]) $\overline{|\alpha|}$ denotes, as usual, the maximum absolute value of the conjugates of an algebraic number α. If α is an algebraic integer, $\overline{|\alpha|}$ is called the size of α.

[2]) The height $H(\alpha)$ of an algebraic number α is defined as the maximum absolute value of the relatively prime integer coefficients in its minimal defining polynomial.

[3]) This obviously holds if $L_{i_{t+1}} = \lambda_{iji_t} L_{i_t}$ with some algebraic number $\lambda_{iji_t} \neq 0$.

[4]) $\mathbf{Z}_K$ and $\mathbf{Z}_L$ denote the rings of integers of K and L, respectively.

Theorem 1 remains valid for any (i.e. not necessarily integer) elements α_{ij}, $\beta(\neq 0)$ of K with heights $H(\alpha_{ij})\leqq\mathscr{A}$ and $H(\beta)\leqq\mathscr{B}$. Then we have

(9) $$\max_{1\leqq i\leqq m}\overline{|x_i|} < \exp\{(mn)^2(8N)^{32(N+1)}(R\log R^*)^2(R+\log\mathscr{A}\mathscr{B})\}$$

and

(10) $$\max_{1\leqq i\leqq m}\overline{|x_i|} < \exp\{(mn)^2(8N)^{30(N+1)}D(\log D)^{3N-1}(D^{1/2}+\log\mathscr{A}\mathscr{B})\}$$

for all solutions of (6).

Under the assumptions of Theorem 1 (6) possesses only finitely many solutions in algebraic integers of L and they can be effectively determined.

We remark that FELDMAN [8], [9] also obtained effective results on the equation (6) with certain rather special linear forms and with β replaced by any polynomial $f\in\mathbf{Z}_L[x_1,\dots,x_m]$ of small degree relative to n[5]). Our Theorem 1 contains the special case $f=$constant of Feldman's theorems.

Since the linear factors $x-\alpha_1 y,\dots,x-\alpha_n y$ occurring in (3) satisfy obviously the conditions of our above theorem, from (9) and (10) we get, as a special case of Theorem 1, the following improvement of the above-quoted result of BAKER and COATES [6].

Corollary 1.1. *Let K be an algebraic number field of degree $k\geqq 2$ with discriminant D_K and regulator R_K. Let $\alpha_1,\dots,\alpha_n$ be elements of K with heights $\leqq\mathscr{A}$ such that among them at least three are distinct, and let β be a non-zero element in K with height$\leqq\mathscr{B}$. Then all solutions $(x,y)\in\mathbf{Z}_K^2$ of (3) satisfy*

(11) $$\max(\overline{|x|},\overline{|y|}) < \exp\{4n^2(8k)^{32(k+1)}(R_K\log R_K^*)^2(R_K+\log\mathscr{A}\mathscr{B})\}$$

and

(12) $$\max(\overline{|x|},\overline{|y|}) < \exp\{4n^2(8k)^{30(k+1)}|D_K|(\log|D_K|)^{3k-1}(|D_K|^{1/2}+\log\mathscr{A}\mathscr{B})\},$$

where $R_K^*=\max(R_K,e)$.

(11) and (12) furnishe the best known bounds for the solutions of (3). When K is a totally imaginary quadratic extension of a totally real number field, much sharper estimate has been established in [17] for the solutions of (3) in real algebraic integers x, y of K.

The next corollary includes Baker's theorem on the Thue equation [2].

Corollary 1.2. *Let L be an algebraic number field with degree l and discriminant D_L. Let $F(x,y)\in\mathbf{Z}_L[x,y]$ be a binary form of degree $n\geqq 3$ with $\overline{|F|}\leqq H$ such that $F(1,0)\neq 0$ and $F(x,1)$ has at least three distinct zeros*[6]). *Suppose b is a non-zero algebraic integer in L with $\overline{|b|}\leqq B$ and $|N_{L/Q}(b)|\leqq B^*$. Then all solutions $(x,y)\in\mathbf{Z}_L^2$ of (1) satisfy*

(13) $$\max\overline{|x|},\overline{|y|}) < B^{1/n}\exp\{(8lN)^{36(lN+1)}\big(|D_L|(4H)^{3l(ln-1)}\big)^N\big(\log(4H|D_L|)\big)^{3lN}\times \times[\big(|D_L|(4H)^{3l(ln-1)}\big)^{N/2}+\log B^*]\},$$

where $N=n(n-1)(n-2)$.

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[5]) Further results concerning norm form equations in several unknowns will be referred to in Section 3.

[6]) As usual, $\overline{|F|}$ denotes the maximum absolute value of the conjugates of the coefficients of a polynomial F with algebraic coefficients.

(7), (8) and (13) are best possible in terms of B. When $L=\mathbf{Q}$, (13) provides Feldman's estimate [10] with an upper bound computed explicitly in terms of each parameter. Following a more direct deduction (cf. [2], [10], [29], [33]), from the recent estimates for linear forms in the logarithms of algebraic numbers one can derive better bounds for the solutions of (1) in terms of H. See SPRINDŽUK [29] and STARK [33]. Their bounds*) can be further improved by using the recent estimates of BAKER [5] and VAN DER POORTEN and LOXTON [23], see [21].

3. Explicit bounds for the integer solutions of norm form equations

As before, let L and K be algebraic number fields such that $L\subset K$, $L\neq K$ and consider the norm form equation

$$N_{K/L}(\alpha_1 x_1+\ldots+\alpha_m x_m) = \beta \tag{14}$$

in algebraic integers $x_1, \ldots, x_m$ of L, where $0\neq\beta\in L$ and $\alpha_1, \ldots, \alpha_m$ are linearly independent elements of K over L.

First consider the classical case $L=\mathbf{Q}$. The $\mathbf{Z}$-module $M=\{\alpha_1, \ldots, \alpha_m\}$ is called degenerate (see [7], p. 299), if the vector space V over $\mathbf{Q}$ which is generated by M contains a subspace V' such that $V'=\lambda K'$ for some $\lambda\in K$ and some subfield $K'\subset K$, where K' is neither $\mathbf{Q}$ nor an imaginary quadratic field. If M is degenerate, there are β's for which (14) has infinitely many rational integer solutions ([7], [25]). On the other hand, if M is non-degenerate, by a well-known theorem of SCHMIDT [25] (14) has only finitely many rational integer solutions for any fixed β. However, Schmidt's proof is ineffective, that is it does not provide any algorithm for determining all the solutions or deciding the solvability of (14). For $m=2$ such an algorithm was earlier established by BAKER [2]. For $m=3$ and for any K which is a totally imaginary quadratic extension of a totally real number field GYŐRY and LOVÁSZ ([19]; see also [12], Théorème 6) obtained such an algorithm. Further, effective bounds for the solutions of relatively special norm form equations in three unknowns were given by SKOLEM [27], [28], BAKER [1] and FELDMAN [9]. As mentioned in Section 2, in [8] Feldman obtained effective results on norm form equations in an arbitrary number of unknowns, but with certain rather special algebraic numbers $\alpha_1, \ldots, \alpha_m$.

We return now to the general case when in (14) L is any proper subfield of K and $m\geqq 2$ is an arbitrary integer. Under certain assumptions made on $\alpha_1, \ldots, \alpha_m$, Theorem 1 enables us to get explicit upper bounds for the sizes of the solutions $(x_1, \ldots, x_m)\in\mathbf{Z}_L^m$ of (14).

If $\alpha_1, \ldots, \alpha_m\in K$, denote by $L^{(i)}(\mathbf{x})=\alpha_1^{(i)}x_1+\ldots+\alpha_m^{(i)}x_m$ $(1\leqq i\leqq n)$ the conjugates of the form $L(\mathbf{x})=\alpha_1 x_1+\ldots+\alpha_m x_m$ over L, where $n=[K\colon L]$. *The numbers $\alpha_1, \ldots, \alpha_m$ will be called $\triangle$-connected with respect to K/L if the system of linear forms $L^{(1)}, \ldots, L^{(n)}$ is $\triangle$-connected* (c.f. Section 2). If this is the case, then any finite subset of the $\mathbf{Z}_L$-module $\{\alpha_1, \ldots, \alpha_m\}$ is $\triangle$-connected. Therefore we may say that *a finitely generated $\mathbf{Z}_L$-module M in K is $\triangle$-connected with respect to K/L if it has a $\triangle$-connected system of generators with respect to K/L.*

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*) *Added in proof.* For recent improvements and p-adic generalizations see KOTOV and SPRINDŽUK (*Izv. Akad. Nauk SSSR*, **41** (1977), 723—751).

The following theorem is a consequence of Theorem 1.

Theorem 2. *Let L be an algebraic number field of degree l, K an extension of degree $n\geqq 3$ of L, and D_K the absolute value of the discriminant of K. Suppose $\alpha_1, \dots, \alpha_m$ are $m\geqq 2$ linearly independent elements of K over L with heights $\leqq \mathcal{A}$, such that they are $\triangle$-connected with respect to K/L. Let β be any non-zero element in L with height $\leqq \mathcal{B}$. Then all solutions $(x_1, \dots, x_m)\in \mathbf{Z}_L^m$ of (14) satisfy*

$$\max_{1\leqq i\leqq m} \overline{|x_i|} < \exp\{(8lN)^{32(lN+1)} D_K^{3N/n} (\log D_K)^{3lN-1} (D_K^{3N/2n} + \log \mathcal{A}\mathcal{B})\}, \tag{15}$$

where $N=n(n-1)(n-2)$.

In the case $L=\mathbf{Q}$ Theorem 2 implies that any $\triangle$-connected $\mathbf{Z}$-module of K is non-degenerate. For $\triangle$-connected $\mathbf{Z}$-modules our Theorem 2 makes effective Schmidt's famous theorem [25].

We present only two consequences of Theorem 2, but it is easy to give various other m-tuples of algebraic numbers $\alpha_1, \dots, \alpha_m$ satisfying the conditions of Theorem 2.

Corollary 2.1. *Let L be an algebraic number field of degree l, and let $\alpha_2, \dots, \alpha_m$ be algebraic numbers with degrees $n_i\geqq 3$ $(2\leqq i\leqq m)$ over L and with heights $\leqq \mathcal{A}$ such that for $K=L(\alpha_2, \dots, \alpha_m)$ $[K:L]=n_2\dots n_m=n$. Let N, D_K and β be defined as in Theorem 2 and let $\alpha_1=1$. Then all solutions of (14) in algebraic integers $x_1, x_2, \dots, x_m$ of L satisfy (15).*

Corollary 2.1 again implies Baker's theorem on the Thue equation [2].

Combining Corollary 2.1 with a recent theorem of Schinzel [24] (cf. Lemma 5) we can state explicitly a wide class of m-tuples of algebraic numbers $1, \alpha_2, \dots, \alpha_m$ for which the above assertion holds. The following special case seems to be of particular interest.

Corollary 2.2. *Let $n_2, \dots, n_m$ be positive integers greater than 2 and let b be a non-zero rational integer. Suppose $a_2, \dots, a_m$ are rational integers such that $|a_i|$ is not a d_i-th power with $1<d_i|n_i$ $(2\leqq i\leqq m)$, and for any distinct i, j, $2\leqq i, j\leqq n$, n_i, n_j or a_i, a_j are relatively prime. If $\alpha_i^{n_i}=a_i$ $(2\leqq i\leqq m)$ and $K=\mathbf{Q}(\alpha_2, \dots, \alpha_m)$, then all rational integer solutions $x_1, \dots, x_m$ of the equation*

$$N_{K/Q}(x_1+\alpha_2 x_2+\dots+\alpha_m x_m) = b \tag{16}$$

satisfy

$$\max_{1\leqq i\leqq m} |x_i| < \exp\{(8N)^{35(N+1)} A^{3(N-1)} (\log A)^{3N} (A^{3(N-1)/2} + \log|b|)\}, \tag{17}$$

where $A=|a_2\dots a_m|$, $n=n_2\dots n_m$ and $N=n(n-1)(n-2)$.

Our next theorem can be regarded as a generalization of Theorem 2 which corresponds to the case $s=1$. For simplicity Theorem 3 will be stated only for algebraic integers $\alpha_1, \dots, \alpha_m$.

Theorem 3. *Let $L=K_0\subset K_1\subset \dots \subset K_s=K$ be a sequence of algebraic number fields with $[K_i: K_{i-1}]\geqq 3$ for $i=1, \dots, s$, $[L:\mathbf{Q}]=l$, $[K: L]=n$, and D_K the discriminant of K. Let $M_i\subset \mathbf{Z}_{K_i}$ be a $\mathbf{Z}_L$-module with generators of size $\leqq A'$ which are linearly independent over K_{i-1} and are $\triangle$-connected with respect to K_i/K_{i-1} $(i=1, \dots, s)$. Let further β be an integer in L with size at most B. If $\alpha_1, \dots, \alpha_m$ are* *linearly independent elements of* $M_1 \dots M_s$ [7]) *over* L *with* $\overline{|\alpha_j|} \leqq A$ *for* $j=1, \dots, m$, *then for any solution* $(x_1, \dots, x_m) \in \mathbf{Z}_L^m$ *of* (14)

(18)

$$\max_{1 \leqq i \leqq m} \overline{|x_i|} < A^{ml-1} \exp\{2ln(8lN)^{32(lN+1)} |D_K|^{3N/n} (\log |D_K|)^{3lN-1} (|D_K|^{3N/2n} + \log A'B)\}$$

holds, where $N=n(n-1)(n-2)$.

We state now such a consequence of this theorem which cannot be deduced from Theorem 2.

Corollary. *Let* $n \geqq 3$ *and* a *be rational integers such that* $|a|$ *is not a d-th power with* $1<d|n$, *and let* $K=\mathbf{Q}(\sqrt[n^{m-1}]{a})$, $\alpha_i=(\sqrt[n^{m-1}]{a})^{n^{i-2}}$ *for* $i=2, \dots, m$. *Given a non-zero rational integer b, all solutions of* (16) *in rational integers* $x_1, \dots, x_m$ *satisfy*

$$\max_{1 \leqq i \leqq m} |x_i| < \exp\{(8N)^{35(N+1)} (|a| \log |a|)^{3N} (|a|^{3N/2} + \log |b|)\}, \tag{19}$$

where $N=n^{m-1}(n^{m-1}-1)(n^{m-1}-2)$.

4. Explicit bounds for the integer solutions of discriminant form equations

Let $L \subset K$ be algebraic number fields with $[K: L]=n \geqq 3$, $\alpha_1, \dots, \alpha_m$ elements of K, and δ a non-zero element in L. Consider the *discriminant form*[8]) *equation*

$$\mathrm{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_m x_m) = \delta \tag{20}$$

in algebraic integers $x_1, \dots, x_m$ of L. If $1, \alpha_1, \dots, \alpha_m$ are linearly dependent over L and (20) is solvable, then it has infinitely many solutions in algebraic integers of L. In what follows we shall suppose that $1, \alpha_1, \dots, \alpha_m$ are linearly independent over L. Further, we may assume without loss of generality that $\alpha_1, \dots, \alpha_m$ are algebraic integers in K.

When $L=\mathbf{Q}$, (20) was investigated by many authors in various special cases. For references see our paper [20]. In a recent paper of the first named author [13] (see also [14]) effective bounds have been established for the size of the integer solutions of (20). In other words, in the case $L=\mathbf{Q}$ (20) has only finitely many solutions in rational integers and these can be effectively determined. Our Theorem 1 enables us to generalize this result to arbitrary algebraic number fields L as follows.

Theorem 4. *Let* $L \subset K$ *be algebraic number fields with degrees* $[L:\mathbf{Q}]=l$, $[K:L]=n \geqq 3$, D_K *the discriminant of K, and* δ *a non-zero integer in L with* $\overline{|\delta|} \leqq d$ *and* $|N_{L/\mathbf{Q}}(\delta)| \leqq d^*$. *Suppose* $\alpha_1, \dots, \alpha_m$ *are algebraic integers in K with sizes at*

[7]) By the product $M_1 \dots M_s$ we understand the $\mathbf{Z}_L$-module generated by the products of the generators of $M_1, \dots, M_s$.

[8]) If $K=L(\alpha_1, \dots, \alpha_m)$, the *discriminant form* $\mathrm{Discr}_{K/L}(\alpha_1 x_1 + \dots + \alpha_m x_m)$ is a decomposable form of degree $n(n-1)$ with coefficients in L, otherwise it is identically vanishing (see e.g. [13], [14] or [20]).

most A such that $1, \alpha_1, \ldots, \alpha_m$ *are linearly independent over L. Then all solutions* $(x_1, \ldots, x_m) \in \mathbf{Z}_L^m$ *of* (20) *satisfy*

$$(21)\qquad \max_{1 \leqq i \leqq m} \overline{|x_i|} < d^{\frac{1}{n(n-1)}} \exp\left\{(8N)^{32(N+1)} |D_K|^{\frac{3N}{ln}} (\log |D_K|^{3N-1}\left(|D_K|^{\frac{3N}{2ln}} + \log Ad^*\right)\right\},$$

where $N = ln\,(n-1)(n-2)$.

It is easily seen that if $K = L(\alpha_1, \ldots, \alpha_m)$ (i.e. if the discriminant form is not identically vanishing), then in (21) $|D_K|$ can be estimated from above by $D_L^n (2A)^{m(ln)^2}$, D_L being here the absolute value of the discriminant of L.

Using certain recent effective theorems on algebraic numbers with given discriminant [15], [16], in [20] we sharpen (21) and give a $\mathfrak{p}$-adic generalization of Theorem 4.

Finally we remark that from Theorem 4 one can deduce Theorem 3B of [15] on algebraic integers with given relative discriminant, but only with a weaker estimate than that of Theorem 3B in [15].

5. Preliminary results

To prove our theorems we need some lemmas. The proof of Lemma 1 is based on a recent explicit estimate of VAN DER POORTEN and LOXTON [23] for linear forms in the logarithms of algebraic numbers.

Lemma 1 (GYŐRY [18]). *Let K be an algebraic number field with degree k, discriminant* D_K *and regulator* R_K. *Let r denote the number of fundamental units of K. Suppose* $\gamma_1, \gamma_2, \gamma_3, x_1, x_2, x_3$ *are non-zero algebraic integers in K with* $\overline{|\gamma_i|} \leqq G$, $|N_{K/Q}(x_i)| \leqq N$ $(1 \leqq i \leqq 3)$, *satisfying*

$$\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = 0.$$

Then there exists a unit ε *in K such that*

$$(22)\qquad \max_{1 \leqq i \leqq 3} \overline{|x_i \varepsilon|} < \exp\left\{(16(r+3)k)^{16r+30} (R_K \log R_K^*)^2 (R_K + \log (GN))\right\}$$

and

$$(23)\qquad \max_{1 \leqq i \leqq 3} \overline{|x_i \varepsilon|} < \exp\left\{c_1 |D_K| (\log |D_K|)^{3k-1} (|D_K|^{1/2} + \log (GN))\right\},$$

where $R_K^* = \max(R_K, e)$ *and* $c_1 = \dfrac{64^k}{k^{k-3}} (16(r+3)k)^{15(r+2)}$.

Lemma 1 improves and generalizes Lemma 4 of [11] and Lemma 3 of [13].

In proving our assertions we shall use some known properties of the heights and the sizes of algebraic numbers (see e.g. [2], [29], [35]). Further, we shall need the inequality

$$(24)\qquad \overline{|\alpha/\beta|} \leqq \overline{|\alpha|} \cdot \overline{|\beta|}^{l-1},$$

where α is any algebraic number, β is any non-zero algebraic integer and α/β lies in a number field L of degree l. (24) follows at once from

$$\overline{|\alpha/\beta|}^{[K:L]} = \overline{|N_{K/L}(\alpha)/N_{K/L}(\beta)|} \leqq \overline{|N_{K/L}(\alpha)|}\,\overline{|N_{K/L}(\beta)|}^{l-1} \leqq (\overline{|\alpha|}\,\overline{|\beta|}^{l-1})^{[K:L]},$$

K being here any number field containing α, β and L.

Lemma 2. *Let $L \subset K$ be algebraic number fields with $[L:\mathbf{Q}]=l$, $\alpha_{ij} \in \mathbf{Z}_K$ with $\overline{|\alpha_{ij}|} \leqq A$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ and $\beta_i \in K$ with $\overline{|\beta_i|} \leqq B$ $(1 \leqq i \leqq n)$. Suppose that $x_1, \ldots, x_m$ is the only solution in L of the system of equations*

$$\sum_{j=1}^{m} \alpha_{ij} x_j = \beta_i \quad (1 \leqq i \leqq n). \tag{25}$$

Then we have

$$\max_{1 \leqq j \leqq m} \overline{|x_j|} \leqq m^{(ml+1)/2} A^{ml-1} B. \tag{26}$$

Proof. Consider the linear equation system consisting of all those equations which are conjugate over L to the equations occurring in (25). $x_1, \ldots, x_m$ satisfy this system of equations too. We shall show that this extended system of equations has no other solutions in the field $\mathbf{C}$ of complex numbers.

Let $[K:L]=k$, and let $\omega_1, \ldots, \omega_k$ be a basis for the extension K/L. Write $\alpha_{ij} = \sum_{p=1}^{k} \gamma_{ijp} \omega_p$ with $\gamma_{ijp} \in L$. Denote by $K^{(1)}=K, \ldots, K^{(k)}$ the conjugate fields of K over L and by $\alpha^{(1)}=\alpha, \ldots, \alpha^{(k)}$ the corresponding conjugates of any element α of K. Consider the $n \times m$ matrix $\mathcal{A}=(\alpha_{ij})$, the $nk \times m$ matrices $\mathcal{A}'=(\alpha'_{ij})$, $\mathcal{A}''=(\alpha''_{ij})$, and the $nk \times nk$ matrix $\mathcal{W}=(w_{ij})$, where $\alpha'_{ij}=\alpha_{pj}^{(q)}$, $\alpha''_{ij}=\gamma_{pjq}$ for $i=k(p-1)+q$, $p=1, \ldots, n$, $q=1, \ldots, k$, $j=1, \ldots, m$, and $w_{ij}=\omega_s^{(q)}$ for $i=kr+q$, $q=1, \ldots, k$, $j=kr+s$, $s=1, \ldots, k$, $r=0, \ldots, n-1$ and $w_{ij}=0$ otherwise. By the assumption the column rank of $\mathcal{A}$ relative to L is m, so the column rank of $\mathcal{A}''$ relative to L is also m. Since the entries of $\mathcal{A}''$ belong to L, its rank relative to $\mathbf{C}$ is m. $\mathcal{W}$ being non-singular, $\mathcal{A}'=\mathcal{W}\mathcal{A}''$ is also of rank m relative to $\mathbf{C}$ which was to be proved.

We can now choose m equations, say $\sum_{j=1}^{m} \alpha'_{i_h j} x_j = \beta'_{i_h}$ $(h=1, \ldots, m)$, from the extended equation system so that $\det(\alpha_{i_h j}) \neq 0$. Solving this latter equation system by Cramer's rule, by virtue of (24) and the Hadamard's determinant inequality we get (26).

We remark that rank $\mathcal{A}'=m$ implies $m \leqq nk$. Furthermore, it is easy to verify that under the assumptions of the lemma even $m \leqq nN$ holds, where $[L(\alpha_{i_1}, \ldots, \alpha_{i_m}):L] \leqq N$ for each i with $1 \leqq i \leqq m$.

Lemma 3 (Stark [34]). *Let $K_1, \ldots, K_m$ be algebraic number fields with discriminants $D_{K_1}, \ldots, D_{K_m}$ and let $K=K_1 \ldots K_m$ with discriminant D_K. Suppose $[K_i:\mathbf{Q}]=k_i$ and let $k=[K:\mathbf{Q}]$. Then*

$$D_K \,\Big|\, \prod_{i=1}^{m} D_{K_i}^{k/k_i}.$$

Lemma 4. *Let L be an algebraic number field and let $K_1, \ldots, K_m$ be extensions of L with $[K_i:L]=n_i \geqq 3$ $(1 \leqq i \leqq m)$. Let M_i be a finitely generated $\mathbf{Z}_L$-module in K_i and suppose that it is $\triangle$-connected with respect to K_i/L $(1 \leqq i \leqq m)$. Suppose $[K:L]=n_1 \ldots n_m$, where $K=K_1 \ldots K_m$. Then the $\mathbf{Z}_L$-module $M_1 \ldots M_m$ is $\triangle$-connected with respect to K/L.*

Proof. We prove the assertion only for $m=2$. Then the general case easily follows by induction on m.

Let α_1 and α_2 be primitive elements of K_1/L and K_2/L, respectively. Denote the conjugates of α_1 and α_2 over L by $\alpha_1^{(1)}=\alpha, \ldots, \alpha^{(n_1)}$ and $\alpha_2^{(1)}=\alpha_2, \ldots, \alpha_2^{(n_2)}$. Then $K=L(\alpha_1, \alpha_2)$ has $n_1 n_2$ conjugate fields over L and each conjugate field K' of K over L is uniquely determined by the isomorphisms $\alpha_1 \to \alpha_1^{(i)}$, $\alpha_2 \to \alpha_2^{(j)}$ for which $K'=L(\alpha_1^{(i)}, \alpha_2^{(j)})$, that is by a pair of indices (i, j) with $1 \leqq i \leqq n_1$, $1 \leqq j \leqq n_2$.

Let us fix some finite systems of generators of M_1 and M_2 and consider the linear form $l(\mathbf{x})$ whose coefficients are the products of the generators of M_1 and M_2. We may denote the conjugates of l over L by $l^{(i,j)}$ $(1 \leqq i \leqq n_1,\ 1 \leqq j \leqq n_2)$. By the assumption made on M_1 and M_2 both $l^{(1,j)}, \ldots, l^{(n_1,j)}$ and $l^{(i,1)}, \ldots, l^{(i,n_2)}$ are $\triangle$-connected systems for every i, j with $1 \leqq i \leqq n_1$, $1 \leqq j \leqq n_2$. So the system of all conjugates of l over L is also $\triangle$-connected. Consequently $M_1 M_2$ has a $\triangle$-connected system of generators, that is $M_1 M_2$ is $\triangle$-connected with respect to K/L.

Lemma 5. (SCHINZEL [24]). *Let L be any field and let $n_1, \ldots, n_m$ be positive integers. Assume that the characteristic of L does not divide $n_1 \ldots n_m$ and $\alpha_i^{n_i}=a_i \in L^*$ $(=L \setminus 0)$ for $i=1, \ldots, m$. $[L(\alpha_1, \ldots, \alpha_m): L]=n_1 \ldots n_m$ if and only if for all primes p $\prod_{p|n_i} a_i^{x_i}=g^p$ implies $x_i \equiv 0 \pmod p\ (p\,|\,n_i)$ and $\prod_{2|n_i} a_i^{x_i}=-4g^4$, $n_i x_i \equiv 0 \pmod 4$ $(2\,|\,n_i)$ implies $x_i \equiv 0 \pmod 4$ $(2\,|\,n_i)$* [9]).

This general theorem will be used in the special case when L is an algebraic number field.

Lemma 6. *Let $L \subset K$ be algebraic number fields with $[K:\mathbf{Q}]=k \geqq 2$, $[K:L]=n$, and let R_K, D_K and r denote the regulator, the discriminant and the number of fundamental units respectively of K. Suppose $v \in \mathbf{N}$ and $\alpha \in \mathbf{Z}_K$ with $\overline{|N_{K/L}(\alpha)|} \leqq m$. Then there exist a unit ε and an integer β in K such that*

$$\alpha = \varepsilon^v \beta, \quad N_{K/L}(\varepsilon) = 1$$

and

$$\overline{|\beta|} \leqq m^{1/n} \exp\{v c_2 R_K\} \leqq m^{1/n} \exp\{v c_3 |D_K|^{1/2} (\log |D_K|)^{k-1}\},$$

where $c_2=nr\,(31rk^2 \log 6k)^r$ *and* $c_3=2c_2/(k-1)^{k-3}$.

PROOF. Put $|N_{K/Q}(\alpha)|=M$. By a theorem of SIEGEL [26] and by Lemma 3 of [18] there are a unit ε_1 and an integer γ in K with the properties

$$\alpha = \varepsilon_1^{vn}\gamma, \quad \overline{|\gamma|} \leqq M^{1/k} T_1 \leqq M^{1/k} T_2, \quad |\overline{\gamma^{-1}}| \leqq M^{-1/k} T_1 \leqq M^{-1/k} T_2,$$

where $T_1=\exp\{(1/2) v c_2 R_K\}$ and $T_2=\exp\{(1/2) v c_3 |D_K|^{1/2} (\log |D_K|)^{k-1}\}$. Thus for the unit $\varepsilon_2=N_{K/L}(\varepsilon_1)$ lying in L $\varepsilon_2^{vn}=N_{K/L}(\alpha) N_{K/L}(\gamma^{-1})$ holds. It is easy to verify that $\varepsilon=\varepsilon_1^n/\varepsilon_2$ and $\beta=\varepsilon_2^v \gamma$ satisfy the required conditions.

[9]) $(p|n_i)$ means here "for all i such that $p|n_i$".

6. Proofs of the theorems and their corollaries

PROOF OF THEOREM 1. Suppose that (6) is solvable. Then the forms $L_1, \dots, L_n$ are not identically vanishing. Since $\mathscr{L}=\{L_1, \dots, L_n\}$ is a $\triangle$-connected system, there are two forms in $\mathscr{L}$ having a linear combination with non-zero coefficients which belongs to $\mathscr{L}$. We may suppose without loss of generality that L_1, L_2 is such a pair.

Case (i). First suppose that L_1 and L_2 are not proportional and that, for convenience, L_3 can be expressed as a linear combination of L_1 and L_2 with non-zero algebraic coefficients. Let $\Delta_{pq}=\begin{vmatrix}\alpha_{pi} & \alpha_{qi}\\ \alpha_{pj} & \alpha_{qj}\end{vmatrix}$ $(1\leqq p,\, q\leqq 3)$, where i, j are chosen so that $\Delta_{12}\neq 0$. Then for $\lambda_1=\Delta_{23}$, $\lambda_2=\Delta_{31}$, $\lambda_3=\Delta_{12}$

$$(27)\quad \lambda_1 L_1+\lambda_2 L_2+\lambda_3 L_3 = 0,\quad \lambda_1, \lambda_2, \lambda_3 \in \mathbf{Z}_{K_{123}},\quad \lambda_1\lambda_2\lambda_3 \neq 0,\quad \overline{|\lambda_1|}, \overline{|\lambda_2|}, \overline{|\lambda_3|} \leqq 2A^2$$

hold.

Case (ii). If L_1 and L_2 are proportional and e.g. $\alpha_{1j}\neq 0$, then for $\lambda_1=\alpha_{2j}^2$, $\lambda_2=\alpha_{1j}\alpha_{2j}$, $\lambda_3=-2\alpha_{1j}\alpha_{2j}$ (27) also holds with L_3 replaced by L_2.

Let $\boldsymbol{\xi}=(\xi_1, \dots, \xi_m)\in \mathbf{Z}_L^m$ be an arbitrary but fixed solution of (6) and write $\beta_i=L_i(\boldsymbol{\xi})$ $(i=1, \dots, n)$. From (6) we get

$$(28)\qquad \beta_1 \dots \beta_n = \beta.$$

Further, in Case (i) (27) implies

$$\lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3 = 0$$

with the above $\lambda_1, \lambda_2, \lambda_3$, where $\beta_j \in \mathbf{Z}_{K_{123}}$ and

$$|N_{K_{123}/Q}(\beta_j)| \leqq B^{*\frac{[K_{123}(\beta):\, L(\beta)]}{[K_{123}(\beta):\, K_{123}]}} \leqq B^{*N} \quad (1\leqq j\leqq 3).$$

Therefore, by Lemma 1 there exists a unit ε in K_{123} satisfying

$$(29)\qquad \overline{|\varepsilon\beta_j|} \leqq U,\quad j=1, 2,$$

where

$$U = \exp\left\{3N\big(16(N+2)N\big)^{16N+14}(R\log R^*)^2(R+\log AB^*)\right\}.$$

In Case (ii) we can see in the same way that there is a unit ε in K_{122} satisfying (29).

We are now going to show that for every j with $1\leqq j\leqq n$ there are $\varphi_j, \psi_j\in \mathbf{Z}_K$ such that

$$(30)\qquad \overline{|\varphi_j|}, \overline{|\psi_j|} \leqq U^n \quad\text{and}\quad \varepsilon\beta_j = \varphi_j/\psi_j$$

with the above ε. For $j=1, 2$ these are implied by (29). Put $j\geqq 3$. $\mathscr{L}$ being $\triangle$-connected, by definition there is a sequence $L_2=L_{i_1}, L_{i_2}, \dots, L_{i_s}=L_j$ in $\mathscr{L}$ such that for each t, $1\leqq t\leqq s-1$, L_{i_t} and $L_{i_{t+1}}$ have a linear combination, say L_{j_t}, with non-zero algebraic coefficients which belongs to $\mathscr{L}$. Further we may assume $s\leqq n$. It follows now in the same way as above that there exists a unit ε_t in $K_{i_t i_{t+1} j_t}$ with the properties

$$(31)\qquad \varepsilon_t\beta_{i_t} = \gamma_{t,i_t},\quad \varepsilon_t\beta_{i_{t+1}} = \gamma_{t,i_{t+1}},\quad \overline{|\gamma_{t,i_t}|}, \overline{|\gamma_{t,i_{t+1}}|} \leqq U \quad (t=1, \dots, s-1).$$

Writing now $\varepsilon_0=\varepsilon$, $\gamma_{0,i_1}=\varepsilon\beta_2$, we get $\varepsilon_{t-1}/\varepsilon_t=\gamma_{t-1,i_t}/\gamma_{t,i_t}$ for $t=1, \dots, s-1$, whence

$$\varepsilon\beta_j = \frac{\varepsilon_0}{\varepsilon_{s-1}}\gamma_{s-1,i_s} = \left(\prod_{t=1}^{s-1}\frac{\varepsilon_{t-1}}{\varepsilon_t}\right)\gamma_{s-1,i_s} = \left(\prod_{t=1}^{s}\gamma_{t-1,i_t}\right)\Big/\left(\prod_{t=1}^{s-1}\gamma_{t,i_t}\right).$$

By (31) $\varphi_j=\prod_{t=1}^{s}\gamma_{t-1,i_t}$ and $\psi_j=\prod_{t=1}^{s-1}\gamma_{t,i_t}$ satisfy the required conditions.

With the notation $\sigma_j=\varphi_j^{n-1}\prod_{\substack{i=1\\i\neq j}}^{n}\psi_i$, $\varrho_j=\psi_j^{n-1}\prod_{\substack{i=1\\i\neq j}}^{n}\varphi_i$ it follows from (28) and (30)

$$\beta_j^n = \beta\sigma_j/\varrho_j \quad (j=1, \dots, n).$$

β_j is of degree $\leqq N$ over $\mathbf{Q}$ for each j, thus by (24) and (30) we have

$$|\overline{L_j(\xi)}| = |\overline{\beta_j}| \leqq B^{1/n}U^{2nN} \quad (j=1, \dots, n). \tag{32}$$

Since (5) has no non-trivial solutions in L, the only solution in L of the system of equations

$$L_j(\mathbf{x}) = \beta_j \quad (j=1, \dots, n)$$

is $\boldsymbol{\xi}=(\xi_1, \dots, \xi_m)$. Applying Lemma 2 and the estimate $m\leqq nN$ mentioned at the end of its proof, by (32) we obtain

$$\max_{1\leqq i\leqq m}|\overline{\xi_i}| \leqq m^{(ml+1)/2}A^{ml-1}B^{1/n}U^{2nN} < B^{1/n}U^{2mnN}, \tag{33}$$

where $l=[L:\mathbf{Q}]$. This implies (7).

If we use (23) of Lemma 1 in place of (22), we get (29) and (33) with

$$U = \exp\left\{3\frac{64^N}{N^{N-4}}(16(N+2)N)^{15(N+1)}D(\log D)^{3N-1}(D^{1/2}+\log AB^*)\right\}$$

and (8) follows at once.

In order to prove (9) and (10) suppose that α_{ij} are any (not necessarily integer) elements in K. Denote the leading coefficient of the minimal defining polynomial of α_{ij} by a_{ij}. Then $\alpha'_{ij}=\left(\prod_{s=1}^{m}a_{is}\right)\alpha_{ij}$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$ are integers in K with sizes $\leqq 2\mathscr{A}^m$ and, if (6) is solvable in $\mathbf{Z}_L$, $\beta'=\beta\prod_{i=1}^{n}\prod_{s=1}^{m}a_{is}$ is also an integer in K with size $\leqq 2\mathscr{B}\mathscr{A}^{mn}$. Further, (6) is equivalent to the equation

$$\prod_{i=1}^{n}(\alpha'_{i1}x_1+\dots+\alpha'_{im}x_m) = \beta'.$$

Repeating the above proof for this equation with $A=2\mathscr{A}^m$, $B=2\mathscr{B}\mathscr{A}^{mn}$ and $B^*=(2\mathscr{B}\mathscr{A}^{mn})^{[L(\beta):\mathbf{Q}]}$, we can easily derive (9) and (10).

Remark. As it turns out from the above proof, in Theorem 1 it suffices to assume that N, R and D are upper bounds for the degrees, regulators and absolute values of the discriminants of those number fields K_{ijl} which were utilized in the course of the proof.

Proof of Corollary 1.2. Write (1) in the form

$$F(x, y) = a(x-\alpha_1 y)\dots(x-\alpha_n y) = b$$

and suppose that $x, y \in \mathbf{Z}_L$ is a solution of (1). Putting $x'=ax$, $\beta'=a^{n-1}b$ and $\alpha_i'=a\alpha_i$ for $i=1, \dots, n$, we get

$$(x'-\alpha_1' y)\dots(x'-\alpha_n' y) = \beta'$$

and we can apply Theorem 1. It is easy to see that $\overline{|\beta'|} \leqq BH^{n-1}$, $|N_{L/Q}(\beta')| \leqq H^{(n-1)l}B^*$ and α_i' are integers with sizes $<\overline{|a|}+H \leqq 2H$ for every i, $1 \leqq i \leqq n$. The degree l_{ijk} of the field $K_{ijk}=L(\alpha_i, \alpha_j, \alpha_k)$ does not exceed lN $(i, j, k=1, \dots, n)$. Further, by Lemma 3 the absolute value of the discriminant of K_{ijk} can be estimated from above by

$$\left(|D_L|^{1/l}|D_{Q(\alpha_i)/Q}(\alpha_i')|^{1/l_i}|D_{Q(\alpha_j)/Q}(\alpha_j')|^{1/l_j}|D_{Q(\alpha_k)/Q}(\alpha_k')|^{1/l_k}\right)^{l_{ijk}} \leqq \left(|D_L|(4H)^{3l(ln-1)}\right)^N,$$

where $l_s=[\mathbf{Q}(\alpha_s):\mathbf{Q}]$ $(1 \leqq s \leqq n)$. By virtue of (24) $\overline{|x|} \leqq \overline{|x'|}H^{l-1}$, so (13) follows from (8) of Theorem 1.

Proof of Theorem 2. Denote the conjugate fields of K over L by $K^{(1)}, \dots, K^{(n)}$ and let $K_{ijk}=K^{(i)}K^{(j)}K^{(k)}$ $(i, j, k=1, \dots, n)$. By Lemma 3 the absolute values of the discriminants of the fields K_{ijk} do not exceed $D_K^{3(n-1)(n-2)}$ and $[K_{ijk}:\mathbf{Q}] \leqq ln(n-1)(n-2)$, $1 \leqq i, j, k \leqq n$. Since $\alpha_1, \dots, \alpha_m$ are linearly independent over L and are $\triangle$-connected with respect to K/L, we can apply Theorem 1 to (14), that is to the equation

$$\prod_{i=1}^{n} (\alpha_1^{(i)}x_1+\dots+\alpha_m^{(i)}x_m) = \beta.$$

From (10) we get (15) for every solution $(x_1, \dots, x_m) \in \mathbf{Z}_L^m$ of this equation.

Proof of Corollary 2.1. In view of $n_i \geqq 3$ the numbers $1, \alpha_i$ are $\triangle$-connected with respect to $L(\alpha_i)/L$, thus the $\mathbf{Z}_L$-module $M_i=\{1, \alpha_i\}$ is also $\triangle$-connected with respect to $L(\alpha_i)/L$ $(i=2, \dots, m)$. By Lemma 4 the $\mathbf{Z}_L$-module $M_2\dots M_m$ as well as the numbers $1, \alpha_2, \dots, \alpha_m$ in this module are $\triangle$-connected with respect to K/L. Since $\alpha_1=1, \alpha_2, \dots, \alpha_m$ are obviously linearly independent over L, they satisfy all conditions of Theorem 2 and so (15) holds for every solution $(x_1, \dots, x_m) \in \mathbf{Z}_L^m$ of (14).

Proof of Corollary 2.2. By Lemma 5 we have $[K:\mathbf{Q}]=n_2\dots n_m$. The absolute value of the discriminant of α_i is $n_i^{n_i}|a_i|^{n_i-1}$ $(i=2, \dots, m)$, thus by Lemma 3 the absolute value of the discriminant of $K=\mathbf{Q}(\alpha_2, \dots, \alpha_m)$ is not greater than $n^n A^{n\left(1-\frac{1}{N}\right)}$. Applying now Corollary 2.1, we get (17) for every solution of (16) in rational integers $x_1, \dots, x_m$.

Proof of Theorem 3. We first show by induction on s that $\gamma \in M=M_1\dots M_s$ and $N_{K/L}(\gamma)=\beta$ imply

$$(34) \quad \overline{|\gamma|} < (n-1)A' \exp\{ln(8lN)^{32(lN+1)}|D_K|^{3N/n}(\log|D_K|)^{3lN-1}(|D_K|^{3N/2n}+\log A'B)\}.$$

Since the heights of the generators of the $\mathbf{Z}_L$-module M_1 do not exceed $(2A')^{ln}$ and $H(\beta) \leqq (2B)^l$, for $s=1$ (34) follows from Theorem 2. Suppose now that (34) is proved for $s-1$ with $s \geqq 2$. Put $K_{s-1}=K'$, $[K': L]=n_1$, $[K: K']=n_2$ and $M'=M_1 \dots M_{s-1}$. Then we have

$$\beta = N_{K/L}(y) = N_{K'/L}(N_{K/K'}(y)).$$

By Lemma 6 there exist a unit ε and an integer β' in K' such that

$$N_{K/K'}(y) = \varepsilon^{n_2}\beta', \quad N_{K'/L}(\varepsilon) = 1 \tag{35}$$

and $\overline{|\beta'|} \leqq B^{1/n_1} \exp\{c_4 |D_{K'}|^{1/2} (\log |D_{K'}|)^{n_1 l-1}\}$, where $c_4 = 2n_1 n_2 (n_1 l-1) \cdot (31(n_1 l-1)(ln_1)^2 \log 6ln_1)^{ln_1-1}/(ln_1-1)^{ln_1-3}$ and $D_{K'}$ signifies the discriminant of K'. This latter inequality together with $D_{K'}^{n_2} | D_K$ implies

$$|\overline{\beta'}| < B^{1/n_1} \exp\{c_5 |D_K|^{1/2n_2} (\log |D_K|)^{n_1 l-1}\} \tag{36}$$

with $c_5 = (4ln_1)^{3ln_1}$.

By the assumption y can be written as $y = y_1\mu_1 + \dots + y_t\mu_t$ with $y_1, \dots, y_t \in M'$ and $t \leqq n_2$, where $\mu_1, \dots, \mu_t$ are such generators of size $\leqq A'$ of the $\mathbf{Z}_L$-module M which are linearly independent over K' and are $\triangle$-connected with respect to K/K'. Writing $y' = \varepsilon^{-1}y$, $y_i' = \varepsilon^{-1}y_i$ $(1 \leqq i \leqq t)$, (35) implies

$$N_{K/K'}(y_1'\mu_1 + \dots + y_t'\mu_t) = N_{K/K'}(y') = \beta'.$$

Applying Theorem 2 to this equation we obtain

$$\max_{1 \leqq i \leqq t} |\overline{y_i'}| < \exp\{c_5 c_6 |D_K|^{\frac{6N_2+1}{2n_2}} (\log |D_K|)^{n_1 l(3N_2+1)-2} (|D_K|^{\frac{3N_2-1}{2n_2}} + \log A'B) = B', \tag{37}$$

where $c_6 = ln_1(8ln_1 N_2)^{32(ln_1 N_2+1)}$ and $N_2 = n_2(n_2-1)(n_2-2)$. It follows from (35) and (37) that $N_{K'/L}(y_i) = N_{K'/L}(y_i') = \beta_i$ with $|\overline{\beta_i}| \leqq (B')^{n_1}$ $(1 \leqq i \leqq t)$. By the induction hypothesis $y_i \in M'$ yields

$$\overline{|y_i|} < (n_1-1)A' \exp\{c_7 |D_{K'}|^{3N_1/n_1} (\log |D_{K'}|)^{3lN_1-1} (|D_{K'}|^{3N_1/2n_1} + \log A'B'^{n_1})\} <$$
$$< \exp\{n_1 c_5 c_6 c_7 |D_K|^{3N/n} (\log |D_K|)^{3lN-1} (|D_K|^{3N/2n} + \log A'B)\} = T,$$

where $N_1 = n_1(n_1-1)(n_1-2)$ and $c_7 = ln_1(8lN_1)^{32(lN_1+1)}$. Consequently $\overline{|y|} \leqq n_2 A' T$, whence, in view of $n_1 c_5 c_6 c_7 < ln(8lN)^{32(lN+1)}$, (34) immediately follows.

Suppose now that $(x_1, \dots, x_m) \in \mathbf{Z}_L^m$ is an arbitrary solution of (14). Writing $\alpha_1 x_1 + \dots + \alpha_m x_m = y$, by virtue of the linear independence of $\alpha_1, \dots, \alpha_m$ over L we can apply Lemma 2 and from (34) we get (18).

PROOF OF THE COROLLARY TO THEOREM 3. Let $K_0 = \mathbf{Q}$, let $K_i = \mathbf{Q}(\alpha_{m-i+1})$, and let M_i be the $\mathbf{Z}$-module in $\mathbf{Z}_{K_i}$ generated by $1, \alpha_{m-i+1}$ $(i = 1, \dots, m-1)$. By Lemma 5 $[K: \mathbf{Q}] = [K_{m-1}: \mathbf{Q}] = n^{m-1}$, so $[K_i: K_{i-1}] = n \geqq 3$ for any i with $1 \leqq i \leqq m-1$, that is α_{m-i+1} is of degree n over K_{i-1} for $i = 1, \dots, m-1$. Consequently, $1, \alpha_{m-i+1}$ are linearly independent over K_{i-1}, $\overline{|\alpha_{m-i+1}|} \leqq |a|^{1/n}$ and M_i is $\triangle$-connected with respect to K_i/K_{i-1} for $i = 1, \dots, m-1$. Since $\alpha_1 = 1, \alpha_2, \dots, \alpha_m$ are linearly independent elements of $M_1 \dots M_{m-1}$ over $\mathbf{Q}$, we can apply Theorem 3 with $L = \mathbf{Q}$. The absolute value of the discriminant of K can be estimated by the absolute value $n^{(m-1)n^{m-1}} a^{n^{m-1}-1}$ of the discriminant of $\sqrt[n^{m-1}]{a}$. Thus from (18) we get (19) for every solution $(x_1, \dots, x_m) \in \mathbf{Z}^m$ of (16).

PROOF OF THEOREM 4. Denote by $K^{(1)}, \dots, K^{(n)}$ the conjugate fields of K over L. Let $L(\mathbf{x})=\alpha_1 x_1+\dots+\alpha_m x_m$ and let $L^{(1)}(\mathbf{x}), \dots, L^{(n)}(\mathbf{x})$ be the conjugates of $L(\mathbf{x})$ over L. Put

$$l_{ij}(\mathbf{x}) = L^{(i)}(\mathbf{x})-L^{(j)}(\mathbf{x}) = (\alpha_1^{(i)}-\alpha_1^{(j)})x_1+\dots+(\alpha_m^{(i)}-\alpha_m^{(j)})x_m \quad (1 \leqq i,\ j \leqq n).$$

Suppose that (20) is solvable. Then $l_{ij}(\mathbf{x})\not\equiv 0$ for any $i\neq j$ and so $K=L(\alpha_1,\dots,\alpha_m)$. Since for any $l_{ij}(\mathbf{x}), l_{i'j'}(\mathbf{x})$ with $(i, j)\neq(i', j')$ and $i\neq j,\ i'\neq j'$

$$l_{ij}(\mathbf{x})+l_{ji'}(\mathbf{x})+l_{i'i}(\mathbf{x}) = 0, \quad l_{ji'}(\mathbf{x})+l_{i'j'}(\mathbf{x})+l_{j'j}(\mathbf{x}) = 0$$

if $i'\neq j$ and

$$l_{ij}(\mathbf{x})+l_{i'j'}(\mathbf{x})+l_{j'i}(\mathbf{x}) = 0$$

if $i'=j$, the system of linear forms $l_{ij}(\mathbf{x})$ with $i\neq j$, $1\leqq i, j\leqq n$ is $\triangle$-connected. Further, the system of equations

$$l_{ij}(\mathbf{x}) = 0 \quad (i \neq j,\ i,j = 1, \dots, n)$$

is equivalent to the system of equations $L^{(1)}(\mathbf{x})=\dots=L^{(n)}(\mathbf{x})$. Because of the linear independence of $1, \alpha_1, \dots, \alpha_m$ over L this latter system of equations has no solutions $\mathbf{x}\neq 0$ with components in L. Thus the equation

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n} l_{ij}(\mathbf{x}) = (-1)^{n(n-1)/2}\delta$$

which is equivalent to (20) satisfies all conditions of Theorem 1. Furthermore, in the course of the above arguments we used the linear dependence of the linear forms $l_{uv}(\mathbf{x}),\ l_{vw}(\mathbf{x}),\ l_{wu}(\mathbf{x})$ $(u\neq v,\ 1\leqq u, v, w\leqq n)$ only. Therefore, by the Remark following the proof of Theorem 1 it suffices to estimate from above the absolute values of the discriminants and the degrees of the fields $K_{uvw}=K^{(u)}K^{(v)}K^{(w)}$ containing the coefficients of $l_{uv},\ l_{vw},\ l_{wu}$. Hence, with the notation of Theorem 1, we may choose $N=ln(n-1)(n-2)$ and $D=|D_K|^{3N/ln}$ and (21) follows from (8).

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