

On a Q -recurrent Finsler space of second order

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1. Introduction

Let F_n be an n -dimensional Finsler space equipped with positively homogeneous metric function $F(x, \dot{x})$ of degree one in its directional arguments and satisfies the requisite conditions [1]¹⁾ imposed upon it. The fundamental metric tensor of the space is given by

$$(1.1) \quad g_{ij}(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2} \partial_i \partial_j F^2(x, \dot{x}), \quad (\partial_i \equiv \partial/\partial \dot{x}^i).$$

The projective covariant derivative [2] of a vector field $X^i(x, \dot{x})$ with respect to x^k is given by

$$(1.2) \quad \dot{x}_{((k)}^i = \partial_k x^i - (\partial_m x^i) \Pi_{rk}^m \dot{x}^r + x^h \Pi_{hk}^i$$

where the projective connection coefficients

$$(1.3) \quad \Pi_{hk}^i(x, \dot{x}) \stackrel{\text{def}}{=} G_{hk}^i - \frac{1}{(n+1)} (2\delta_{(h}^i G_{k)r}^r + \dot{x}^i G_{rkh}^r)$$

are positively homogeneous of degree zero in $\dot{x}^i$'s and satisfy the following relations:

$$(1.4) \quad \text{a) } \Pi_{hkr}^i \dot{x}^h = 0 \quad \text{and} \quad \text{b) } \partial_h \Pi_{jk}^i = \Pi_{hjk}^i.$$

In particular the projective covariant derivative of x^i vanishes i.e.

$$(1.5) \quad \dot{x}_{((k)}^i = 0.$$

The commutation formulae involving the projective covariant derivative of tensor field $T_j^i(x, \dot{x})$ are given by

$$(1.6) \quad \partial_h (T_{j((k)}^i) - (\partial_h T_j^i)_{((k)}) = T_j^s \Pi_{shk}^i - T_s^i \Pi_{jhk}^s$$

and

$$(1.7) \quad 2T_{j[(h)(k)]}^i = -(\partial_r T_j^i) Q_{shk}^r \dot{x}^s + T_j^s Q_{shk}^i - T_s^i Q_{jhk}^s,$$

where the projective entity $Q_{hjk}^i(x, \dot{x})$ is given by

$$(1.8) \quad Q_{hjk}^i(x, \dot{x}) = 2 \{ \partial_{[k} \Pi_{j]h}^i - (\partial_r \Pi_{h[j}^i) \Pi_{k]s}^r \dot{x}^s + \Pi_{h[j}^i \Pi_{k]r}^i \}$$

¹⁾ The numbers in square brackets refer to the references given at the end of the paper.

and satisfies the following relation [2]

$$(1.9) \quad Q_{hk}^i = Q_{jhk}^i \dot{x}^j.$$

The first order recurrency condition for the projective entity $Q_{hjk}^i(x, \dot{x})$ is given by

$$(1.10) \quad Q_{hjk(l)}^i = u_l Q_{hjk}^i,$$

where $u_l(x)$ is a recurrence vector depending only upon positional coordinates.

Transvecting (1.10) by $\dot{x}^h$ in view of equations (1.5) and (1.9), we get

$$(1.11) \quad Q_{jk(l)}^i = u_l Q_{jk}^i.$$

2. Q -recurrent Finsler space of second order

Definition (2.1): In an Fn if the projective Q_{hjk}^i satisfies the relations

$$(2.1) \quad Q_{hjk(l)(m)}^i = b_{lm} Q_{hjk}^i$$

and

$$(2.2) \quad Q_{hjk}^i \neq 0,$$

where $b_{lm}(x, \dot{x})$ is a tensor field depending both upon positional and directional arguments, then it is said to be Q -recurrent Finsler space of second order and b_{sm} is called a recurrent tensor field. In this paper we shall denote such a Finsler space by Fn^* .

Theorem (2.1). A relation between $u_l(x)$ and $b_{lm}(x, \dot{x})$ is given by

$$(2.3) \quad b_{lm} = u_{l((m))} + u_l u_m.$$

PROOF. Differentiating (1.10) projectively covariantly with respect to x^m and using equations (1.10) itself and (2.1), we get the required result (2.3).

Theorem (2.2). In an Fn^* the recurrence tensor field $b_{lm}(x, \dot{x})$ is non-symmetric.

PROOF. Commutating (2.1) with respect to the indices s and m we get

$$(2.4) \quad 2Q_{hjk[(l)(m)]}^i = (b_{lm} - b_{ml})Q_{hjk}^i.$$

In view of the commutation formula (1.7) the above equation reduces to

$$(2.5) \quad \begin{aligned} (b_{lm} - b_{ml})Q_{hjk}^i = & -(\partial_r Q_{hjk}^i)Q_{lm}^r + Q_{hjk}^r Q_{rlm}^i - \\ & - Q_{rjk}^i Q_{hlm}^r - Q_{hrk}^i Q_{jlm}^r - Q_{hjr}^i Q_{klm}^r \end{aligned}$$

which proves the theorem.

Theorem (2.3). In an Fn^* if u_s is independent of $\dot{x}^i$ the recurrence tensor field $b_{sm}(x, \dot{x})$ satisfies the following relation

$$(2.6) \quad \dot{x}^s \{(b_{lm} - b_{ml})_{(s)} - u_s(b_{lm} - b_{ml})\} = 0.$$

PROOF. In view of the commutation formula (1.6) differentiating (2.5) projective covariantly with respect to x^s and using the equations (1.10), (1.11), (2.5) itself and the fact that u_s is independent of $\dot{x}^i$, we have

$$(2.7) \quad (b_{lm} - b_{ml})_{((s))} Q_{hjk}^i = u_s (b_{lm} - b_{ml}) Q_{hjk}^i + \\ + Q_{lm}^r \{ Q_{hjk}^p \Pi_{prs}^i - Q_{pjk}^i \Pi_{hrs}^p - Q_{hpk}^i \Pi_{jrs}^p - Q_{hjp}^i \Pi_{krs}^p \}.$$

Transvecting (2.7) by $\dot{x}^s$ and noting equation (1.4a), we obtain

$$(2.8) \quad Q_{hjk}^i \{ (b_{lm} - b_{ml})_{((s))} - u_s (b_{lm} - b_{ml}) \} \dot{x}^s = 0$$

which in view of the assumption that $Q_{hjk}^i \neq 0$, gives the required result.

Theorem (2.4). *In an Fn^* the recurrence tensor field satisfies the following relation:*

$$(2.9) \quad \dot{x}^s \{ (b_{lm} - b_{ml})_{((s))((n))} - u_{s((n))} (b_{lm} - b_{ml}) - 2(b_{lm} - b_{ml})_{[(n)]} u_s \}.$$

PROOF. Differentiating (2.7) projective covariantly with respect to x^n and using (1.10), (1.11), (2.3) and (2.7) itself, we get

$$(2.10) \quad Q_{hjk}^i [(b_{lm} - b_{ml})_{((s))((n))} - u_{s((n))} (b_{lm} - b_{ml}) - 2(b_{lm} - b_{ml})_{[(n)]} u_s] - \\ - Q_{lm}^r \{ Q_{hjk}^p \Pi_{prs((n))}^i - Q_{pjk}^i \Pi_{hrs((n))}^p - Q_{hpk}^i \Pi_{jrs((n))}^p - Q_{hjp}^i \Pi_{krs((n))}^p \} = 0.$$

Transvecting (2.10) by $\dot{x}^s$ and noting the equations (1.4a) and (1.5), we get

$$(2.11) \quad Q_{hjk}^i [(b_{lm} - b_{ml})_{((s))((n))} - u_{s((n))} (b_{lm} - b_{ml}) - 2(b_{lm} - b_{ml})_{[(n)]} u_s] \dot{x}^s = 0$$

which in view of the assumption (2.2) yields the required result.

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