

Divisibility properties of arithmetical functions

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1. Let us suppose that $g_1(n), \dots, g_k(n)$ are additive arithmetical functions having only non-negative integer values. Let $\underline{m}$ denote vectors the components m_i of which are non-negative integers. Let

$$\underline{g}(n) = (g_1(n), \dots, g_k(n)).$$

The purpose of this paper is to calculate the asymptotical density of the sequence n satisfying the relation $\underline{g}(n) = \underline{m}$, for fixed $\underline{m}$'s, and for a large class of functions.

Let

$$(1.1) \quad N(x, \underline{m}) = \sum_{\substack{n \leq x \\ \underline{g}(n) = \underline{m}}} 1.$$

Let $\mathbf{P}$ denote the set of primes, and for every feasible $\underline{m}$ let $\mathbf{P}_{\underline{m}}$ be the set of primes p satisfying the relation $\underline{g}(p) = \underline{m}$. Let $\pi_{\underline{m}}(x)$ be the number of elements in $\mathbf{P}_{\underline{m}}$ that do not exceed x . It is obvious that

$$(1.2) \quad \sum_{\underline{m}} \pi_{\underline{m}}(x) = \pi(x).$$

Furthermore we shall suppose that the inequality

$$(1.3) \quad \sum_{\underline{m}} |\pi_{\underline{m}}(x) - c_{\underline{m}} \pi(x)| \leq \frac{Kx}{(\log x)^{1+\gamma}}$$

holds, where K and γ are positive constants, $c_{\underline{m}} \geq 0$ for every $\underline{m}$.

From (1.2) we have easily that

$$(1.4) \quad \sum_{\underline{m}} c_{\underline{m}} = 1.$$

On the assumption of (1.3) we can determine the asymptotical behaviour of $N(x, \underline{m})$.

Let

$$(1.5) \quad G_{\underline{m}}(s) = \sum_{\underline{g}(n) = \underline{m}} \frac{1}{n^s},$$

s being a complex variable. Let

$$(1.6) \quad e_{\underline{m}}(p^l) = \begin{cases} 1 & \text{when } \underline{g}(p^l) = \underline{m}, \\ 0 & \text{otherwise.} \end{cases}$$

We investigate the asymptotic of $G_m(s)$ at $s=1$. Then, by using a tauberian theorem due to H. DELANGE [1] — which we state as Lemma 1 — we get almost immediately the asymptotic of $N(x, \underline{m})$.

2. Lemma 1. (1) If $a_n \geq 0$, b is a real number and for $\operatorname{Re} s > b$ we have

$$(2.1) \quad \sum_{n=1}^{\infty} a_n \cdot n^{-s} = (s-b)^{-a} \sum_{j=0}^m h_j(s) \log^j (1/(s-b))$$

where a is positive, $h_0(s), \dots, h_m(s)$ are regular for $\operatorname{Re} s \geq b$ and $h_m(b) \neq 0$, then

$$(2.2) \quad \sum_{n \leq x} a_n \sim b^{-1} h_m(b) \Gamma^{-1}(a) x^b (\log \log x)^m (\log x)^{a-1}.$$

(2) If $a_n \geq 0$ and

$$\sum_{n=1}^{\infty} a_n \cdot n^{-s} = h_0(s)(s-1)^{-a} + \sum_{j=1}^m h_j(s)(s-1)^{-c_j} + h(s)$$

for $\operatorname{Re} s > 1$, where a is a real number not equal to zero or a negative integer, $h_0(s), \dots, h_m(s), h(s)$ are regular for $\operatorname{Re} s \geq 1$, $h_0(1) \neq 0$ and the constants c_j are real numbers not equal to zero or a negative integer all less than a , then

$$\sum_{n \leq x} a_n \sim h_0(1) \Gamma^{-1}(a) x (\log x)^{-1}.$$

(3) If $a_n \geq 0$ and

$$\sum_{n=1}^{\infty} a_n \cdot n^{-s} = \sum_{j=0}^m h_j(s) \log^j (1/(s-1))$$

for $\operatorname{Re} s > 1$, where $h_0(s), \dots, h_m(s)$ are regular for $\operatorname{Re} s > 1$, $m \geq 1$ and $h_m(1) \neq 0$, then

$$\sum_{n \leq x} a_n \sim m h_m(1) x (\log \log x)^{m-1} (\log x)^{-1}.$$

3. The case $\underline{m}=0$.

It is obvious that

$$(3.1) \quad G_0(s) = \prod_p \left(1 + \sum_{l=1}^{\infty} e_0(p^l) p^{-ls} \right),$$

and the product converges absolutely in the halfplane $\operatorname{Re} s > 1$.

a. The case $c_0 > 0$. We shall prove that the function

$$(3.2) \quad u_0(s) = G_0(s) \cdot [\zeta(s)]^{-c_0}$$

is regular in the halfplane $\operatorname{Re} s \geq 1$, and that $u_0(1) \neq 0$.

For this, we take

$$\log u_0(s) = \log G_0(s) - c_0 \log \zeta(s) = \sum_{p \in P_0} \frac{e_0(p)}{p^s} - c_0 \sum_p \frac{1}{p^s} + v_0(s),$$

where $v_0(s)$ is regular and bounded in $\operatorname{Re} s > 3/4$, say. Since

$$\sum_{p \in p_0} \frac{e_0(p)}{p^s} = \int_1^\infty u^{-s} d\pi_0(u) = s \int_1^\infty \frac{\pi_0(u)}{u^{s+1}} du,$$

and

$$\sum_{p \in P} 1/p^s = s \int_1^\infty \frac{\pi(u)}{u^{s+1}} du,$$

therefore

$$(3.3) \quad \log u_0(s) = s \int_1^\infty \frac{\pi_0(u) - c_0 \pi(u)}{u^{s+1}} du + v_0(s).$$

So, by (1.3) the last integral is convergent for $\operatorname{Re} s \geq 1$. Since $\zeta(s)(s-1) \rightarrow 1$ for $s \rightarrow 1$, we get

$$(3.4) \quad G_0(s) = h_0(s)(s-1)^{-c_0}$$

where

$$(3.5) \quad h_0(s) = \frac{G_0(s)}{(s-1)^{c_0}} \left(\frac{s-1}{\zeta(s)} \right)^{c_0}.$$

By Lemma 1, we get

$$(3.6) \quad N(x, 0) \sim A(0) \Gamma(c_0)^{-1} x (\log x)^{c_0-1}$$

where $A(0)$ being a positive constant defined by

$$(3.7) \quad A(0) = \lim_{s \rightarrow 1+0} G_0(s) \cdot (s-1)^{-c_0}.$$

b. The case $c_0=0$.

In this case we can prove only the inequality

$$(3.8) \quad N(x, 0) \ll x \exp \left(\sum_{p < x} \frac{e_0(p)-1}{p} \right),$$

which by

$$\sum_p \frac{e_0(p)}{p} < \infty$$

gives that

$$(3.9) \quad N(x, 0) \ll x / \log x.$$

(3.8) is a special case of a general well known theorem.

4. The case $m \neq 0$.

For a general natural number K , let $G_0(s|K)$ be defined as

$$(4.1) \quad G_0(s|K) = \sum_{\substack{(n,K)=1 \\ g(n)=0}} 1/n^s.$$

Then

$$(4.2) \quad G_0(s|K) = \prod_{p \nmid K} \left(1 + \sum_{l=1}^{\infty} \frac{e_0(p^l)}{p^{ls}} \right).$$

Hence

$$(4.3) \quad G_0(s|K) = A(s|K) \cdot G_0(s),$$

where

$$(4.4) \quad A(s|K) = \prod_{p|K} \left(1 + \sum_{l=1}^{\infty} \frac{e_0(p^l)}{p^{ls}} \right)^{-1}.$$

Let B denote the set of those integers K , for every exact prime power divisor $(p^\alpha || K)$ of which $g(p^\alpha) \neq 0$.

Thus we get the relation

$$(4.5) \quad G_m(s) = \sum_{\substack{K \in B \\ g(K) = m}} \frac{1}{K^s} G_0(s|K).$$

Hence, by (4.2) we get

$$(4.6) \quad G_m(s) = G_0(s) \cdot H_m(s),$$

where

$$(4.7) \quad H_m(s) = \sum_{\substack{K \in B \\ g(K) = m}} \frac{1}{K^s} A(s|K).$$

We can suppose that $g(K) = m$ is soluble for at least one $K \in B$, since in the opposite case $N(x, m) = 0$, and $G_m(s) \equiv 0$.

For every $K \in B$ let

$$(4.8) \quad K = L_1 L_2 R, \quad L = L_1 L_2,$$

where L_1 contains exactly those exact prime power divisors $p^\alpha || K$ for which $\alpha \geq 2$ (if it is an empty set then $L_1 = 1$), L_2 contains those primes $p || K$ for which $c_{g(p)} = 0$ (in (1.3)) and R contains the other prime divisors of K . This representation of K is unique.

Taking into account that $A(s/n)$ is a multiplicative function of n , we get

$$(4.9) \quad A(s|K) = A(s|L) \cdot A(s|R).$$

Let L, L_1, L_2 be the set of the integers that occur as L, L_1, L_2 , respectively. From (4.9) we have

$$(4.10) \quad H_m(s) = \sum_{L \in L} \frac{A(s|L)}{L^s} B(s|L; m - g(L)),$$

where

$$(4.11) \quad B(s|L; r) = \sum \frac{1}{R^s} \cdot A(s|R),$$

the sum is extended over those R for which $R \in B, (R, L) = 1$ and $g(R) = r$.

Let D denote the set of those vectors $\underline{m}$ for which $c_{\underline{m}} > 0$. Let $R(r)$ denote the set of the solutions of the equation

$$(4.12) \quad l_1 \cdot r_1 + \dots + l_t \cdot r_t = r,$$

where l_i are positive integers, $r_i \in D, r_i \neq 0$ ($i = 1, \dots, t$).

Let $S=(r_1, \dots, r_t, l_1, \dots, l_t)$ denote a particular solution of (4.12). We say that an R in the right hand side of (4.11) belongs to S , when after a suitable permutation of prime-factors of $R=p_1 \dots p_s$ ($s=l_1+l_2+\dots+l_t$) we get

$$(4.13) \quad \begin{cases} \underline{g}(p_1) = \dots = \underline{g}(p_{l_1}) = r_1 \\ \underline{g}(p_{l_1+1}) = \dots = \underline{g}(p_{l_1+l_2}) = r_2 \\ \vdots \end{cases}$$

Then, from (4.11) we have

$$(4.14) \quad B(s/L; r) = \sum_{S \in R(r)} \Delta(s/L, S),$$

where

$$(4.15) \quad \Delta(s/L, S) = \sum \frac{A(s/R)}{R^s}$$

and the sum is extended over those R for which $R \in B$, $(R, L)=1$ and $\underline{g}(R) \in S$.

Let $\mathbf{T}$ be an arbitrary set of primes. We have

$$(4.16) \quad \prod_{p \in \mathbf{T}} \left(1 + \frac{z A(s/p)}{p^s} \right) = \exp \left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s) \right),$$

where

$$(4.17) \quad a_l(s) = \sum_{p \in \mathbf{T}} \frac{A(s/p)^l}{p^{ls}}.$$

Let

$$(4.18) \quad H_n(s, \mathbf{T}) = \sum_{\substack{p_1 < \dots < p_n \\ p_i \in \mathbf{T}}} \frac{A(s/p_1) \dots A(s/p_n)}{(p_1 \dots p_n)^s}.$$

Taking into account that the functions in (4.16) can be expanded in Taylor series of z , comparing the coefficients we get

$$(4.19) \quad \begin{aligned} H_n(s, \mathbf{T}) &= \text{coeff}_{z^n} \exp \left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s) \right) = \\ &= \text{coeff}_{z^n} \left\{ \sum_{v=0}^{\infty} \frac{1}{v!} \left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l a_l(s) \right)^v \right\} = \frac{1}{n!} a_1^n(s) + \sum_{j=0}^{n-1} h_j(s) \cdot a_1^j(s) + h(s) \end{aligned}$$

where $h_j(s)$ ($j=0, \dots, n-1$) and $h(s)$ are bounded and regular functions in the halfplane $\text{Re } s > 3/4$. The bound does not depend on T .

Let now $\mathbf{T}_i$ be the set of those primes p for which $\underline{g}(p)=r_i$, $p \nmid L$.

Then we get

$$(4.20) \quad \sum_{p \in \mathbf{T}_i} \frac{A(s/p)}{p^s} = \sum_{p \in \mathbf{T}_i} 1/p^s + v_1(s) = \sum_{p \in P_{r_i}} 1/p^s + v_1(s) - \sum_{\substack{p \mid L_1 \\ \underline{g}(p)=r_i}} 1/p^s,$$

where $v_1(s)$ is bounded and regular in $\text{Re } s > 3/4$.

First we observe that

$$(4.21) \quad \sum_{p \in P_{\underline{r}_i}} 1/p^s = \int_0^\infty \frac{d\pi_{\underline{r}_i}(u)}{u^s} = c_{\underline{r}_i} \log \frac{1}{s-1} + s \int_0^\infty \frac{\pi_{\underline{r}_i}(u) - c_{\underline{r}_i} \pi(u)}{u^{s+1}} du$$

and that the last integral is absolutely convergent in $\operatorname{Re} s \geq 1$ see (1.3). We estimate the last sum in (4.20). Suppose that $\operatorname{Re} s \geq 1$. Then

$$\sum_{\substack{p/L_1 \\ g(\underline{p})=\underline{r}_i}} 1/p^s \leq \sum_{p/L_1} 1/p \leq \log \prod_{p/L_1} \frac{1}{1-\frac{1}{p}} = c(L_1).$$

Consequently the left side of (4.20) is equal to

$$(4.22) \quad c_{\underline{r}_i} \log \frac{1}{s-1} + v_2(s) + v(s/L, \underline{r}_i)$$

where $v_2(s)$ and $v(s/L, \underline{r}_i)$ are regular and bounded in $\operatorname{Re} s \geq 1$, $v_2(s)$ does not depend on L , and

$$|v(s/L, \underline{r}_i)| \leq c(L_1).$$

Hence, by (4.19) we get

$$(4.23) \quad \Delta(s|\mathbf{L}, S) = B(S) \left(\log \frac{1}{s-1} \right)^v + \sum_{j=0}^{v-1} t_j(s, L) \left(\log \frac{1}{s-1} \right)^j,$$

where

$$(4.24) \quad B(S) = \prod_{i=1}^t \frac{c_{\underline{r}_i}^i}{l_i!}, \quad v = v(S) = l_1 + \dots + l_t,$$

and

$$|t_j(s, L)| \ll c(L_1)^v \quad \text{in } \operatorname{Re} s \geq 1.$$

For a fixed $\underline{r}$ let

$$(4.25) \quad \mu = \mu(\underline{r}) = \max_S v(S),$$

where $\mu(\underline{r})=0$ when $R(\underline{r})$ is an empty set. The value of μ does not depend on L .

Let

$$(4.26) \quad A_{\underline{m}} = \max_{L_1} \mu(\underline{r} - g(L_1)).$$

Suppose that $A_{\underline{m}} > 0$. Then we get — see (4.23), (4.15), (4.14), (4.10) —

$$(4.27) \quad H_{\underline{m}}(s) = A_{\underline{m}}(s) \left(\log \frac{1}{s-1} \right)^{A_{\underline{m}}} + \sum_{j=0}^{A_{\underline{m}}-1} B_j(s) \left(\log \frac{1}{s-1} \right)^j,$$

where the functions $A_{\underline{m}}(s)$, $B_j(s)$ are regular in $\operatorname{Re} s \geq 1$, and $A_{\underline{m}}(1) \neq 0$.

By (4.6) we get

$$G_{\underline{m}}(s) = (s-1)^{-c_0} \sum_{j=0}^{A_{\underline{m}}} g_j(s) \cdot \left(\log \frac{1}{s-1} \right)^j,$$

$$g_{A_{\underline{m}}}(1) \neq 0.$$

This relation holds for $A_m=0$ too, since we assumed that $\underline{g}(n)=\underline{m}$ has at least one solution. Hence, by Lemma 1 we get:

$$\text{for } c_0 > 0$$

$$(4.28) \quad N(x, \underline{m}) \sim B_m x (\log x)^{-c_0-1} (\log \log x)^{A_m},$$

for $c_0=0$, $A_m > 0$,

$$(4.29) \quad N(x, \underline{m}) \sim B_m x (\log x)^{-1} (\log \log x)^{A_m-1}$$

where B_m is a positive constant.

5. Theorem 1. *On the assumption (1.3), by the notation (4.12), (4.24), (4.25), (4.26) we get:*

(1) *If $c_0 > 0$ and there exists at least one solution of $\underline{g}(n)=\underline{m}$, then*

$$N(x, \underline{m}) \sim B_m x (\log x)^{-c_0-1} (\log \log x)^{A_m}.$$

(2) *If $c_0=0$ and $A_m > 0$, then*

$$N(x, \underline{m}) \sim B_m x (\log x)^{-1} (\log \log x)^{A_m-1},$$

B_m are suitable positive constants.

As it is easy to see from our result follows the assertion due to W. NARKIEWICZ [2] concerning the divisibility properties of integer-valued multiplicative functions defined as values of a polynomial for every prime.

6. Let $f_1(n), \dots, f_k(n)$ be multiplicative functions having positive integer values. Let $q_1, \dots, q_k$ be arbitrary not necessarily distinct prime numbers. We define $g_i(n)$ as the greatest α for which q_i^α is a divisor of $f_i(n)$. It is clear that $g_i(n)$ are additive functions having non-negative integer values.

Assuming that the relation (1.3) holds for

$$\underline{g}(n) = (g_1(n), \dots, g_k(n)),$$

we can use Theorem 1 to determine the asymptotic behaviour of

$$N(x, \underline{m}) = \sum_{\substack{n \leq x \\ \underline{g}(n) = \underline{m}}} 1.$$

To illustrate our theorem we investigate the divisibility properties of the Euler totient function.

Let $q_1 < q_2 < \dots < q_k$ be distinct primes. Let $\underline{m} = (\alpha_1, \alpha_2, \dots, \alpha_k)$, α_i be non-negative integers. Let $D = q_1^{\alpha_1} \dots q_k^{\alpha_k}$. We say that D is a total divisor of N , if D/N and $q_i^{\alpha_i+1} \nmid N$ ($i=1, \dots, k$). We write then $D \parallel N$. Let $N_D(x)$ denote the number of $n \leq x$ satisfying the relation $D \parallel n$.

Let $\pi_{\underline{m}}(x)$ be the number of primes p not exceeding x for which $D \parallel p-1$. Let $Q=q_1 \dots q_k$. We have

$$\pi_{\underline{m}}(x) = \sum_{\substack{p \equiv 1 \pmod D \\ (\frac{p-1}{D}, Q)=1}} 1 = \sum_{\delta|Q} \mu(\delta) \sum_{\substack{p \equiv 1 \pmod{D\delta} \\ p \leq x}} 1 = \sum_{\delta|Q} \mu(\delta) \pi(x, \delta D, 1)$$

where in general $\pi(x, k, l)$ denotes the number of primes $p \leq x$ in the arithmetical progression $l \pmod k$. Using the prime number theorem for arithmetical progression see e.g. K. PRACHAR [3] we get

$$\pi_{\underline{m}}(x) = c_{\underline{m}} lix + O(x/(\log x)^{20})$$

uniformly for $D \leq (\log x)^{10}$, where

$$(6.1) \quad c_{\underline{m}} = \sum_{\delta|Q} \frac{\mu(\delta)}{\varphi(D\delta)}.$$

Taking into account that

$$\pi_{\underline{m}}(x) \ll \frac{lix}{\varphi(D)} \quad \text{for } D \leq \sqrt{x}$$

and

$$\pi_{\underline{m}}(x) \ll \frac{x}{D} \quad \text{for } D \leq x$$

we deduce easily that (1.3) holds. Now

$$(6.2) \quad c_{\underline{0}} = \sum_{\delta|Q} \frac{\mu(\delta)}{\varphi(\delta)} = \prod_{i=1}^k \left(1 - \frac{1}{q_i-1}\right).$$

We see that $c_{\underline{0}} > 0$ if $q_1 > 2$ and $c_{\underline{0}} = 0$ if $q_1 = 2$. Furthermore,

$$(6.3) \quad c_{\underline{m}} = \frac{1}{\varphi(D)} \prod_{i=1}^k \left(1 - \frac{1}{q_i}\right).$$

Thus from theorem 1 we immediately deduce

Theorem 2. (1) If $q_1 > 2$, then

$$N_D(x) \sim B_D x (\log x)^{-c_{\underline{0}}-1} (\log \log x)^{\alpha_1 + \dots + \alpha_k}.$$

(2) If $q_1 = 2$, $\alpha_1 = 0$, then $N_D(x) = 1$ for $x \geq 2$.

(3) If $q_1 = 2$, $\alpha_1 > 1$, then

$$N_D(x) \sim B_D x (\log x)^{-1} (\log \log x)^{\alpha_1-1}.$$

PROOF. Assume that $q_1 > 2$. Then $c_{\underline{r}} > 0$ for every $\underline{r}$, consequently

$$\mu(\underline{r}) = \text{the sum of the components of } \underline{r}$$

(see (4.12), (4.25)). Since

$$A_{\underline{m}} = \max_{L_1 \in \mathbf{L}_1} \mu(\underline{r} - \underline{g}(L_1)),$$

the maximum is taken for $L_1=1$. Thus

$$A_{\underline{m}} = \alpha_1 + \dots + \alpha_k,$$

and we can use Theorem 1.

Assume that $q_1=2$. The assertion (2) is obvious, let $\alpha_1 > 2$. To prove (3) we need take into account that $\underline{c}_r=0$ if and only if the first component of $\underline{r}$ is zero. We have that $A_{\underline{m}}=\alpha_1$ and so we use Theorem 1.

References

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