

On cosine operator functions

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Introduction. Let X be a Banach space over the complex field, $B(X)$ the space of bounded linear operators from X into X , R the real field. A *cosine operator function* is a mapping $C:R \rightarrow B(X)$ such that $C(0)=I$ (the identical operator), for $s, t \in R$

$$(1) \quad C(s+t) + C(s-t) = 2C(s)C(t),$$

and $t \rightarrow t_0$ implies $C(t) \rightarrow C(t_0)$ in the strong operator topology of $B(X)$. For the basic facts on cosine operator functions see e.g. [1], [3] and [4].

Let A be a closed linear operator with domain $D(A)$ dense in X , range $R(A)$ in X and non-void resolvent set $\varrho(A)$. Consider the differential equation

$$(2) \quad u''(t) = Au(t),$$

where $u:R \rightarrow D(A)$. H. FATTORINI [1] has shown that the (generalized) solutions of (2) are closely connected with the cosine operator functions and their indefinite integrals, defined by $T(t)x = \int_0^t C(s)x ds$ ($x \in X$), supposing the Cauchy problem for (2) is uniformly well posed (u.w.p.) in R .

The first part of this paper investigates the behaviour of cosine operator functions and their indefinite integrals at infinity. As a rule the limits $\lim_{t \rightarrow \infty} C(t)$ and $\lim_{t \rightarrow \infty} T(t)$ do not exist even in the uniformly bounded case in any of the standard operator topologies of $B(X)$, therefore, following [2] (Chap. 18), we employ the Cesàro (C_a) and the Abel limits. Some of the results can be applied to study the behaviour of the solutions of (2) under suitable conditions.

In the second part we prove a result characterizing the kernel of an operator $C(b)$ in the range of a cosine operator function.

1.

Definition 1. Suppose $F:(0, \infty) \rightarrow B(X)$ is a strongly measurable operator function such that for $z > 0$, $x \in X$, $e^{-zt}F(t)x$ is Bochner integrable on $(0, \infty)$ relative to Lebesgue measure, and with the notation $A(z)x = z \int_0^\infty e^{-zt}F(t)x dt$ we have $A(z) \in B(X)$. We say that $F(t)$ is weakly (strongly, uniformly) Abel-convergent at infinity and its Abel limit is $P \in B(X)$, if $\lim_{z \rightarrow 0+} A(z) = P$ in the respective operator topologies of $B(X)$.

It is well-known that for every cosine operator function $C(t)$ there are numbers $w \geq 0$, $M(w) \geq 1$ such that on R $\|C(t)\| \leq M(w)e^{w|t|}$. However, in general there is no minimal growth parameter w (cf. [4], Remark on p. 10.).

Definition 2. If $C(t)$ is a cosine operator function, we put $w_0 = \inf \{w \geq 0; e^{-w|t|} \|C(t)\| \text{ is bounded on } R\}$, and call $C(t)$ of type w_0 .

Theorem 1. Suppose the cosine operator function $C(t)$ is of type 0, and for $x \in X$ put $T(t)x = \int_0^t C(s)x ds$ ($t \in R$). Then the following statements are equivalent:

- 1) $T(t)$ is weakly Abel-convergent at infinity,
- 2) $C(t)$ and $T(t)$ are uniformly Abel-convergent at infinity with limits 0,
- 3) there are numbers $v, K > 0$ such that on the interval $(0, v)$ we have $\|zR(z^2; A)\| \leq K$ ($R(u; A)$ will denote the resolvent of the generator operator A of $C(t)$),
- 4) there is a $v > 0$ such that for every $x \in X$ the set $H(x) = \{zR(z^2; A)x; 0 < z < v\}$ is conditionally sequentially weakly compact.

PROOF. 2) clearly implies 1). If 1) holds and $x \in X$, $x^* \in X^*$, then

$$(3) \quad x^* z R(z^2; A)x = x^* z \int_0^\infty e^{-zt} T(t)x dt \rightarrow x^* P x \quad \text{if } z \rightarrow 0+.$$

Now if $v > 0$ and $\{z_n\} \subset (0, v)$ is a sequence, then for some subsequence $\{z_{n_k}\}$ of $\{z_n\}$ we have $z_{n_k} \rightarrow z_0 \in [0, v]$. If $z_0 > 0$ then, by the analyticity of the resolvent for $\operatorname{Re} z > 0$, we obtain $z_{n_k} R(z_{n_k}^2; A)x \rightarrow z_0 R(z_0^2; A)x$, while if $z_0 = 0$, then by (3) we get that $H(x)$ is conditionally sequentially weakly compact, i.e. 4) follows.

If 4) holds, then according to [2] (Theor. 2. 9. 1.) $H(x)$ is bounded, and the principle of uniform boundedness yields 3).

Finally, if 3) is satisfied, then on the interval $(0, v)$ we have $\|z^2 R(z^2; A)\| \leq K \cdot z$, thus $z \rightarrow 0+$ implies $zR(z; A) \rightarrow 0$ in the uniform operator topology of $B(X)$. By [2] (Theor. 18. 8. 1.), $R(z; A)$ is then holomorphic in a neighbourhood of 0, hence in some neighbourhood of 0 $\|R(z^2; A)\|$ is bounded, thus $\|zR(z^2; A)\| \leq N|z|$. Consequently $\lim_{z \rightarrow 0} zR(z^2; A) = 0$ in the uniform operator topology and, since $z \int_0^\infty e^{-zt} C(t)x dt = z^2 R(z^2; A)x$ ($z > 0$, $x \in X$), therefore 2) is true and the proof is complete.

Corollary. Suppose the Cauchy problem for (2) is u.w.p. in R and of type $\leq w$ for every $w > 0$. Then every solution of (2) is Abel-convergent at infinity if and only if 3) or 4) of Theorem 1 is valid. Moreover, then every generalized solution of (2) is Abel-convergent at infinity to 0.

Remark. The Abel convergence of a function $f: (0, \infty) \rightarrow X$ has been defined in [2] (Sec. 18. 2.).

PROOF. Under the given conditions the operator A generates a cosine operator function $C(t)$ of type 0 ([1] (5. 9. Theorem)). If every solution is Abel-convergent, then for $x \in D(A)$

$$zR(z^2; A)x = z \int_0^\infty e^{-zt} T(t)x dt \rightarrow P x \quad (z \rightarrow 0+).$$

Moreover, then $zR(z^2; A)Ax = z(z^2R(z^2; A)x - x) \rightarrow 0$, and if $y \in X$, $u \in \mathcal{D}(A)$, then $x_0 = R(u; A)y \in D(A)$, hence $zR(z^2; A)y = zR(z^2; A)(uI - A)x_0$ converges if $z \rightarrow 0+$. By the uniform boundedness principle, 3) and then 4) of Theorem 1 is valid. The remaining parts of the Corollary are evident, by Theorem 1.

In what follows $R(B)$ and $Z(B)$ denote the range and the zero subspace, respectively of a linear operator B , and $\bar{H}$ denotes the strong closure of the set H .

Theorem 2. *If $C(t)$ is a cosine operator function of type 0, then the following conditions are equivalent:*

- 1) $C(t)$ is weakly Abel-convergent at infinity,
- 2) $C(t)$ is strongly Abel-convergent at infinity,
- 3) for some $v > 0$ and for every $x \in X$ the set $\{zR(z; A)x; 0 < z < v\}$ is conditionally sequentially weakly compact,
- 4) for some $v, K > 0$, $0 < z < v$ implies $\|zR(z; A)\| \leq K$, and $X = \overline{Z(A)} + R(A)$.

Moreover, then the strong Abel limit is a projection operator $P \in B(X)$, for which

- (i) $PC(t) = C(t)P = P$ for $t \in \mathbb{R}$,
- (ii) $APx = 0$ for $x \in X$, $P Ax = 0$ for $x \in D(A)$,
- (iii) $R(P) = Z(A) = \{x \in X; C(s)x = x \text{ for } s \in \mathbb{R}\}$,
- (iv) $Z(P) = \overline{R(A)}$,
- (v) $X = \overline{R(A)} \oplus Z(A)$.

PROOF. If $C(t)$ is of type 0, then for every $z > 0$, $x \in X$ we have

$$A(z)x = z \int_0^\infty e^{-zt} C(t)x dt = z^2 R(z^2; A)x,$$

and

$$(4) \quad \lim_{z \rightarrow 0+} A(z) = \lim_{z \rightarrow 0+} zR(z; A)$$

in the respective topologies. Moreover, then A generates a semigroup of operators $F(t)$, $t \in (0, \infty)$ of class (C_0) and of type ≤ 0 (cf. [1], 5. 11. Remark on p. 92). The Abel-convergence of $F(t)$ is equivalent to the existence of the limit (4). Thus the assertions of the theorem follow from [2], Secs. 18.5.—18.7., while (i) can be proved similarly.

Definition 3. Suppose the operator function $F(t)$ satisfies the conditions occurring in Definition 1, and that for $t, a > 0$, $x \in X$ with the notation

$$C(t, a)x = at^{-a} \int_0^t (t-s)^{a-1} F(s)x ds$$

we have $C(t, a) \in B(X)$. We say that $F(t)$ is weakly (strongly, uniformly) C_a -convergent at infinity and its C_a limit is $Q \in B(X)$, if $\lim_{t \rightarrow \infty} C(t, a) = Q$ in the respective operator topologies of $B(X)$.

Theorem 3. If $C(t)$ is a cosine operator function, for which on R $\|C(t)\| \leq M$, then the following assertions are equivalent:

- 1) $C(t)$ is weakly C_a -convergent at infinity for some $a > 0$,
- 2) $C(t)$ is strongly C_a -convergent at infinity for each $a > 0$,
- 3) $X = \overline{Z(A) + R(A)}$.

Moreover, then the C_a limit is a projection operator $P \in B(X)$ satisfying the assertions of the previous theorem.

PROOF. Under the given conditions for $z > 0$ we have $\|zR(z^2; A)\| \leq \frac{M}{z}$, thus

$$(5) \quad \|zR(z; A)\| \leq M.$$

If 1) holds, then by [2] (Theorem 18.2.1.) $C(t)$ is weakly Abel-convergent and, according to Theorem 2, 3) is true. If 3) is valid, then (5) gives that $C(t)$ is strongly Abel-convergent and, by [2] (Theorem 18.3.3.) 2) holds, while 2) evidently implies 1).

Theorem 4. Suppose that, with the notations of Theorem 1, $\sup \{\|C(t)\|, \|T(t)\|; t \in R\} = M$ is finite. Then for each $a > 0$ $C(t)$ and $T(t)$ are strongly C_a -convergent at infinity to 0.

PROOF. Since $R(z^2; A)x = \int_0^\infty e^{-zt}T(t)x$ ($z > 0$, $x \in X$), thus $\|zR(z^2; A)\| \leq M$ for $z > 0$. By Theorem 1, $C(t)$ and $T(t)$ are strongly Abel-convergent at infinity to 0, and [2] (Theorem 18.3.3.) gives the assertion.

Theorem 5. If the cosine operator function $C(t)$ is continuous with respect to the uniform operator topology of $B(X)$, and $\|C(t)\| \leq M$ on R , then the following statements are equivalent:

- 1) $C(t)$ is uniformly Abel-convergent at infinity to $P \in B(X)$,
- 2) $C(t)$ is uniformly C_a -convergent at infinity to P for each $a > 0$,
- 3) $z=0$ is a simple pole of $R(z; A)$ with residue P .

PROOF. By our previous remarks, it follows from [2], Theorems 18.2.1., 18.3.3. and 18.8.4.

2.

It is known from the spectral theory of cosine operator functions [3] that 0 is an eigenvalue of the operator $C(b)$ ($b \neq 0$) if and only if at least one element of $\left\{-\left[\frac{\pi}{b}\left(k+\frac{1}{2}\right)\right]^2; k \text{ integer}\right\}$ is an eigenvalue of A . The following theorem characterizes the kernel of $C(b)$.

Theorem 6. $x \in X$ is in the kernel of the operator $C(b)$ ($b \neq 0$) in the range of a cosine operator function if and only if the function

$$f(z) = (1 + e^{-2|b|z})zR(z^2; A)x \quad \{z \in \sqrt{Q(A)}\}$$

can be analytically continued to an entire function $h(z)$ for which on the complex plane Z we have $\|h(z)\| \leq Me^{2|b \cdot \operatorname{Re} z|}$.

PROOF. We may and will assume $x \neq 0$, further that $b > 0$, for $C(t)$ is an even function. To prove the only if part, put $X_0 = \{x \in X; C(b)x = 0\}$. Then X_0 is a non-trivial closed subspace, for which $C(s)X_0 \subset X_0$ for $s \in R$. Hence if A is the generator of $C(t)$, then $X_A = X_0 \cap D(A)$ is dense in X_0 , and the restriction of A to X_A is closed with $AX_A \subset X_0$. Suppose $z^2 \in \varrho(A)$ and $x \in X$, then $C(b)R(z^2; A)x = R(z^2; A)C(b)x$, hence $R(z^2; A)X_0 \subset X_A$, and the restriction of $R(z^2; A)$ to X_0 is the unique bounded linear operator in X_0 for which

$$R(z^2; A)(z^2 I - A)x = x \quad (x \in X_A),$$

$$(z^2 I - A)R(z^2; A)x = x \quad (x \in X_0).$$

Put $z_k = (2k+1)\frac{i\pi}{2b}$ (k integer), $H = \{z_k; k \text{ integer}\} \cup \{0\}$, $z \in Z \setminus H$, $x \in X_0$ and

$$U(z)x = z^{-1}(1 + e^{-2bz})^{-1} \int_0^{2b} e^{-zu} C(u)x \, du.$$

Then $U(z)$ is a bounded linear operator in X_0 , and $U(z)Ax = AU(z)x$ for $x \in X_A$. $x \in X_0$ implies

$$C(s+b)x + C(s-b)x = 2C(s)C(b)x = 0,$$

thus on X_0 $C(s+2b) = -C(s)$ for every $s \in R$. Integrating twice by parts, we get for $x \in X_A$

$$U(z)Ax = z^{-1}(1 + e^{-2bz})^{-1} \int_0^{2b} e^{-zu} C''(u)x \, du = -x + z^2 U(z)x.$$

Now if $x \in X_0$, $\{x_n\} \subset X_A$, $x_n \rightarrow x$, then $U(z)x_n \rightarrow U(z)x$ and $AU(z)x_n \rightarrow -x + z^2 U(z)x$, thus the closedness of A implies $AU(z)x = -x + z^2 U(z)x$.

Hence for $z \in \sqrt{\varrho(A)} \cap (Z \setminus H)$ and $x \in X_0$ we get $R(z^2; A)x = U(z)x$. If $x \in X_0$ is fixed, then the function $(1 + e^{-2bz})zU(z)x$ is an analytical continuation on $Z \setminus H$ of $f(z) = (1 + e^{-2bz})zR(z^2; A)x$, consequently $h(z) = \int_0^{2b} e^{-zu} C(u)x \, du$ is the analytical continuation on Z of $f(z)$. Moreover,

$$\|h(z)\| \leq e^{2b|\operatorname{Re} z|} \int_0^{2b} \|C(u)x\| \, du = Me^{2b|\operatorname{Re} z|},$$

which was to be proved.

On the other hand, suppose that $x \in X$ and $f(z) = (1 + e^{-2bz})zR(z^2; A)x$ $\{z \in \sqrt{\varrho(A)}\}$ can be continued to the function $h(z)$ with the stated properties. Since for some $w \geq 0$ $\operatorname{Re} z > w$ implies $zR(z^2; A)x = \int_0^\infty e^{-zu} C(u)x \, du$, therefore with the notation $C_2(u)x = \int_0^u \int_0^v C(t)x \, dt \, dv$ we get for $r > w$, $u > 0$ (see [2], (6.3.9))

$$C_2(u)x = (2\pi i)^{-1} \int_{r-i\infty}^{r+i\infty} e^{zu} h(z) (1 + e^{-2bz})^{-1} z^{-2} \, dz,$$

for the last integral converges absolutely, by the properties of $h(z)$. Calculating residues, it can be shown that with the notation $h^* = \frac{d}{dz}[h(z)(1+e^{-2bz})^{-1}]_{z=0}$ we get for $u > 2b$

$$C_2(u)x = h^* + u \frac{h(0)}{2} + \sum_{k=-\infty}^{\infty} e^z k^u h(z_k) (2bz_k^2)^{-1}.$$

Hence $C_2(u+2b)x + C_2(u)x = 2h^* + (u+b)h(0)$ and, differentiating twice, we obtain $C(u+2b)x + C(u)x = 0$. By (1), it follows $C(u+b)C(b)x = 0$ ($u > 2b$) and, again by (1), with $v > 3b$ we get

$$C(b)x = 2C(v)^2 C(b)x - C(2v) C(b)x = 0,$$

and the proof is complete.

References

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