

Inversion of block Toeplitz and Hankel matrices

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1. Introduction. W. TRENCH [1] derived an algorithm for inverting a hermitian Toeplitz matrix T_n with $O(n^2)$ operations rather than $O(n^3)$ as required by standard matrix inversion methods, and stated a similar algorithm for the nonhermitian case. ZOHAR [2] derived the extended algorithm in detail, and AKAIKE [3] generalized Zohar's derivation to block Toeplitz matrices.

TRENCH [4] modified his algorithm for the case when T_n is a Toeplitz band matrix, by which we mean that it has the form

$$\begin{bmatrix} a_0 & a_{-1} & \dots & a_{-p} & 0 & & \\ a_1 & a_0 & & & a_{-p} & & 0 \\ \vdots & & & & & \ddots & \\ a_q & \cdot & q_{q-1} & & & & \cdot \\ 0 & \cdot & \cdot & a_q & & & a_{-p} \\ & 0 & & \cdot & \cdot & & \\ & & & & a_q & a_q & a_0 \end{bmatrix}.$$

The simplification yields a recursive method for computing the first row and column of the inverse of this n th order Toeplitz band matrix with $O(n)$ operations.

Our purpose is to generalize Trench's algorithm for block matrices. Since the multiplication of the block-elements in general is not a commutative operation, we could not apply Trench's method without any change.

2. Notations and definitions Greek small letters denote square matrices of fixed ($l \times l$, say) types, ι denotes the identity and σ the zero matrix among them. Let

$$(2.1) \quad T_n = \begin{bmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \dots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & & & \varphi_{-n+1} \\ & \varphi_n & \varphi_{n-1} & \varphi_{n-2} & \dots & \varphi_0 \end{bmatrix}$$

denote the Toeplitz matrix which we shall investigate. Let

$$(2.2) \quad H_n = \begin{bmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \\ \vdots & & & \\ \alpha_n & \alpha_{n+1} & \dots & \alpha_{2n} \end{bmatrix}$$

be a Hankel matrix.

We define the block-transpose, the block-symmetry and block-persymmetry for matrices and especially for vectorials.

The block-transpose B^t of a block matrix

$$B = [\beta_{j,k}]$$

is defined as

$$B^t = [\beta_{k,j}].$$

A block-matrix B is block-symmetric, if $B=B^t$. Let J denote the matrix

$$J = \begin{bmatrix} \sigma & \sigma & \dots & \sigma & 1 \\ \sigma & \sigma & \dots & 1 & \sigma \\ \vdots & & \ddots & & \\ \sigma & 1 & \dots & & \sigma \\ 1 & \sigma & \dots & & \sigma \end{bmatrix}.$$

It is obvious that for a matrix

$$B = \begin{bmatrix} \beta_{0,0} & \beta_{0,1} & \dots & \beta_{0,n} \\ \beta_{1,0} & \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & & & \\ \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n} \end{bmatrix}$$

we have

$$BJ = \begin{bmatrix} \beta_{0,n} & \beta_{0,n-1} & \dots & \beta_{0,0} \\ \beta_{1,n} & \beta_{1,n-1} & & \beta_{1,0} \\ \vdots & & & \\ \beta_{n,n} & \beta_{n,n-1} & & \beta_{n,0} \end{bmatrix}$$

and

$$JB = \begin{bmatrix} \beta_{n,0} & \beta_{n,1} & \dots & \beta_{n,n} \\ \beta_{n-1,0} & \beta_{n-1,1} & \dots & \beta_{n-1,n} \\ \vdots & \vdots & & \vdots \\ \beta_{0,0} & \beta_{0,1} & \dots & \beta_{0,n} \end{bmatrix}$$

i.e. in BJ the rows of B and in JB the columns of B are reversed. Consequently

$$JBJ = \begin{bmatrix} \beta_{n,n} & \beta_{n,n-1} & \dots & \beta_{n,0} \\ \beta_{n-1,n} & \beta_{n-1,n-1} & \dots & \beta_{n-1,0} \\ \vdots & \vdots & & \vdots \\ \beta_{n,0} & \beta_{n-1,0} & \dots & \beta_{0,0} \end{bmatrix}.$$

Definition. B is block-persymmetric if $JBJ=B^t$.

Note that all block-Toeplitz-matrices are persymmetric. If H_n is a block Hankel matrix — see (2.2) — then

$$H_n J = \begin{bmatrix} \alpha_n & \alpha_{n-1} & \dots & \alpha_0 \\ \alpha_{n+1} & \alpha_n & \dots & \alpha_1 \\ \vdots & \vdots & & \vdots \\ \alpha_{2n} & \alpha_{2n-1} & \dots & \alpha_n \end{bmatrix}$$

is a block-Toeplitz matrix.

We observe that

$$J = J^t = J^{-1}.$$

Furthermore, it is clear that

$$J \begin{bmatrix} \gamma_0 \\ \vdots \\ \gamma_n \end{bmatrix} = \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_0 \end{bmatrix}$$

and

$$[\gamma_0, \dots, \gamma_n]J = [\gamma_n, \dots, \gamma_0].$$

3. Recursion formulas for two vectorials

Let

$$(3.1) \quad T_m = [\varphi_{r-s}]_{r,s=0}^m$$

be defined for $m=0, 1, \dots, n$. They are principal block minors of T_n .

We assume that T_m are regular.

For the sake of brevity we use the notation;

$$(3.2) \quad f_{m,+}^t = [\varphi_1, \dots, \varphi_{m+1}]; \quad f_{m,-}^t = [\varphi_{-(m+1)}, \dots, \varphi_{-1}],$$

$$(3.3) \quad s_m = T_m^{-1}f_{m,+}; \quad r_m = T_m^{+1}f_{m,-}.$$

Let the block-elements of s_m and r_m denote by $\psi_{v,m}$, $\varrho_{v,m}$ respectively, i.e.

$$(3.4) \quad s_m^t = [\psi_{0,m}, \dots, \psi_{m,m}]; \quad r_m^t = [\varrho_{0,m}, \dots, \varrho_{m,m}].$$

We shall give a recursion formula for the computation of s_m and r_m . Suppose that s_m and r_m are known.

We can write T_{m+1} in two hyper-block forms;

$$(3.5) \quad T_{m+1} = \left[\begin{array}{c|c} T_m & f_{m,-}^t \\ \hline f_{m,+}^t & J \end{array} \middle| \begin{array}{c} \varphi_0 \\ \varphi_0 \end{array} \right] =$$

$$(3.6) \quad = \left[\begin{array}{c|c} \varphi_0 & f_{m,-}^t \\ \hline f_{m,+}^t & T_m \end{array} \right].$$

From (3.3) we get

$$T_{m+1}s_{m+1} = f_{m+1,+}.$$

Hence, by (3.5) we get

$$(3.7) \quad T_m \begin{bmatrix} \psi_{0,m+1} \\ \vdots \\ \psi_{m,m+1} \end{bmatrix} + f_{m,-}^t \cdot \psi_{m+1,m+1} = f_{m,+},$$

$$(3.8) \quad (f_{m,+}^t J) \begin{bmatrix} \psi_{0,m+1} \\ \vdots \\ \psi_{m,m+1} \end{bmatrix} + \varphi_0 \cdot \psi_{m+1,m+1} = \varphi_{m+2}.$$

Multiplying by T_m^{-1} in (3.7), we get

$$(3.9) \quad \begin{bmatrix} \psi_{0,m+1} \\ \vdots \\ \psi_{m,m+1} \end{bmatrix} = s_m - r_m \cdot \psi_{m+1,m+1}.$$

We substitute this into (3.8) and get

$$(3.10) \quad (\varphi_0 - (f_{m,+}^t + J)r_m)\psi_{m+1,m+1} = \varphi_{m+2} - (f_{m,+}^t + J)s_m.$$

From the regularity of T_{m+1} it follows that the equation $T_{m+1}s_{m+1} = f_{m+1,+}$ has a unique solution, therefore the solution of (3.10) (concerning $\psi_{m+1,m+1}$) is unique. Hence the matrix

$$(3.11) \quad \varkappa_{m+1} = \varphi_0 - (f_{m,+}^t + J)r_m$$

is invertible and

$$(3.12) \quad \psi_{m+1,m+1} = \varkappa_{m+1}^{-1} \left(\varphi_{m+2} - (f_{m,+}^t + J) \begin{bmatrix} \psi_{0,m} \\ \vdots \\ \psi_{m,m} \end{bmatrix} \right).$$

(3.9) and (3.12) are an explicite algorithm for the computation of s_{m+1} .

For the computation of r_{m+1} we use (3.6). Since $T_{m-1} \cdot r_{m+1} = f_{m+1,-}$, we have

$$(3.13) \quad \varphi_0 q_{0,m+1} + (f_{m,-}^t - J) \begin{bmatrix} q_{1,m+1} \\ \vdots \\ q_{m+1,m+1} \end{bmatrix} = \varphi_{-m+2},$$

$$(3.14) \quad f_{m,+} q_{0,m+1} + T_m \begin{bmatrix} q_{1,m+1} \\ \vdots \\ q_{m+1,m+1} \end{bmatrix} = f_{m,-}.$$

From the last equation, multiplying it by T_m^{-1} , we deduce

$$(3.15) \quad \begin{bmatrix} q_{1,m+1} \\ \vdots \\ q_{m+1,m+1} \end{bmatrix} = r_m - s_m \cdot q_{0,m+1}.$$

Substituting this into (3.13), we get easily that

$$(3.16) \quad q_{0,m+1} = \lambda_{m+1}^{-1} (\varphi_{-(m+2)} - (f_{m,-}^t - J)r_m),$$

where

$$(3.17) \quad \lambda_{m+1} = \varphi_0 - (f_{m,-}^t, J)s_m.$$

4. The same vectorials for the transpose of T_m

For a general matrix A let A^T denote the transpose of it. It is obvious that

$$(4.1) \quad T_m^T = \begin{bmatrix} \varphi_0^T & \varphi_1^T & \cdots & \varphi_m^T \\ \varphi_{-1}^T & \varphi_0^T & \cdots & \varphi_{m-1}^T \\ \vdots & \vdots & & \vdots \\ \varphi_m^T & \varphi_{m-1}^T & \cdots & \varphi_0^T \end{bmatrix}$$

which is a block Toeplitz matrix, too.

Let $g_{m,-}$ and $g_{m,+}$ be defined by

$$(4.2) \quad g_{m,-}^t = [\varphi_{-1}^T, \dots, \varphi_{-(m+1)}^T]; \quad g_{m,+}^t = [\varphi_{m+1}^T, \dots, \varphi_1^T].$$

Let x_m and y_m be defined from

$$(4.3) \quad T_m^T x_m = g_{m,-}; \quad T_m^T y_m = g_{m,+}.$$

Let $\Theta_{v,m}$ and $\tau_{v,m}$ denote the block elements of x_m and y_m , respectively, i.e.

$$(4.4) \quad x_m^t = [\Theta_{0,m}, \dots, \Theta_{m,m}]; \quad y_m^t = [\tau_{0,m}, \dots, \tau_{m,m}].$$

To give recursion formulas for x_m and y_m we need to write into the formulas of § 3 φ_{-j}^T instead of φ_j , $\Theta_{j,m}$ instead of $\psi_{j,m}$, $\tau_{j,m}$ instead of $\varrho_{j,m}$ and we get

$$(4.5) \quad \begin{bmatrix} \Theta_{0,m+1} \\ \vdots \\ \Theta_{m,m+1} \end{bmatrix} = x_m - y_m \cdot \Theta_{m+1,m+1},$$

where

$$(4.6) \quad \begin{aligned} \Theta_{m+1,m+1} &= \varepsilon_{m+1}^{-1} (\varphi_{-(m+2)}^T - (g_{m,-}^t - J) x_m), \\ \varepsilon_{m+1} &= \varphi_0^T - (g_{m,-}^t - J) y_m. \end{aligned}$$

Similarly we have

$$(4.7) \quad \begin{bmatrix} \tau_{1,m+1} \\ \vdots \\ \tau_{m+1,m+1} \end{bmatrix} = y_m - x_m \cdot \tau_{0,m+1},$$

$$(4.8) \quad \begin{aligned} \tau_{0,m+1} &= \delta_{m+1}^{-1} (\varphi_{m+2}^T - (g_{m,+}^t - J) y_m), \\ \delta_{m+1} &= \varphi_0^T - (g_{m,+}^t - J) x_m. \end{aligned}$$

5. Computation of the inverse of T_n

Let

$$(5.1) \quad T_n^{-1} = [\beta_{rs}]_{r,s=0}^n,$$

and

$$(5.2) \quad (T_n^T)^{-1} = [\gamma_{rs}]_{r,s=0}^n.$$

First of all we observe that

$$(5.3) \quad \beta_{sr}^T = \gamma_{rs} \quad (r, s = 0, 1, \dots, n).$$

First we calculate the first column and row of T_n^{-1} .

$$(5.4) \quad \left[\begin{array}{c|c} \varphi_0 & \varphi_{-1} \dots \varphi_{-n} \\ \hline \varphi_1 & \\ \vdots & \\ \varphi_n & \end{array} \middle| T_{n-1} \right] \begin{bmatrix} \beta_{00} \\ \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma \end{bmatrix}.$$

Hence

$$(5.5) \quad \varphi_0 \cdot \beta_{00} + [\varphi_{-1}, \dots, \varphi_{-n}] \begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = 1$$

$$(5.6) \quad \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \beta_{00} + T_{n-1} \begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \end{bmatrix}.$$

Observing that T_{n-1} is non-singular, and that $T_{n-1}s_{n-1} = f_{n-1,1}$, from the last equation — multiplying it by T_{n-1}^{-1} — we get

$$(5.7) \quad \begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = -s_{n-1}\beta_{0,0}.$$

Substituting it into (5.5) we deduce that

$$(5.8) \quad \begin{aligned} \beta_{00} &= \alpha^{-1}, \\ \alpha &= \varphi_0 - (f_{n-1}^t, -J)s_{n-1}. \end{aligned}$$

In the same way we get

$$(5.9) \quad \left[\begin{array}{c|c} \varphi_0^T & \varphi_1^T \dots \varphi_n^T \\ \hline \varphi_{-1}^T & \\ \vdots & \\ \varphi_{-n}^T & \end{array} \middle| T_{n-1}^T \right] \begin{bmatrix} \gamma_{00} \\ \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \varphi_{-1}^T \\ \vdots \\ \varphi_{-n}^T \end{bmatrix} \gamma_{00} + T_{n-1}^T \begin{bmatrix} \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \end{bmatrix},$$

and multiplying by $(T_{n-1}^T)^{-1}$ we have

$$(5.10) \quad \begin{bmatrix} \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = -x_{n-1} \cdot \gamma_{00}.$$

We need not calculate γ_{00} , since $\gamma_{00} = \beta_{00}^T$. From (5.10) and (5.3) we get

$$(5.11) \quad \beta_{0r} = -\beta_{00} \cdot \Theta_{r-1, n-1}^T \quad (r = 1, 2, \dots, n).$$

Now we consider two consecutive columns in T_n^{-1} . Let k_s denote the block vector the $(s+1)$ 'th block of which is l and all of the others are zero, i.e.

$$k_s = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \\ l \\ \sigma \\ \vdots \\ \sigma \end{bmatrix} \begin{matrix} \rightarrow 0\text{-th place} \\ \\ \\ \rightarrow (s+1)\text{-th place.} \\ \end{matrix}$$

If we take

$$k_s = \begin{bmatrix} \sigma \\ l \end{bmatrix},$$

then

$$k_{s-1} = \begin{bmatrix} l \\ \sigma \end{bmatrix}.$$

We have

$$(5.12) \quad \left[\begin{array}{c|c} T_{n-1} & \begin{bmatrix} \varphi_{-n} \\ \varphi_{-1} \end{bmatrix} \\ \hline \varphi_n \dots \varphi_1 & \varphi_0 \end{array} \right] \begin{bmatrix} \beta_{0,s-1} \\ \beta_{n-1,s-1} \\ \beta_{n,s-1} \end{bmatrix} = \begin{bmatrix} l \\ 0 \end{bmatrix}$$

and that

$$(5.13) \quad \left[\begin{array}{c|c} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} & \begin{bmatrix} \varphi_{-1} \dots \varphi_{-n} \\ T_{n-1} \end{bmatrix} \\ \hline & \end{array} \right] \begin{bmatrix} \beta_{0s} \\ \beta_{1s} \\ \vdots \\ \beta_{ns} \end{bmatrix} = \begin{bmatrix} 0 \\ l \end{bmatrix}.$$

Therefore, from (5.12) and (5.13) we get respectively, that

$$\begin{aligned} T_{n-1} \begin{bmatrix} \beta_{0,s-1} \\ \vdots \\ \beta_{n-1,s-1} \end{bmatrix} + \begin{bmatrix} \varphi_{-n} \\ \varphi_{-1} \end{bmatrix} \beta_{n,s-1} &= l, \\ \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \beta_{0s} + T_{n-1} \begin{bmatrix} \beta_{1s} \\ \vdots \\ \beta_{ns} \end{bmatrix} &= l. \end{aligned}$$

Subtracting these equations

$$T_{n-1} \begin{bmatrix} \beta_{1s} - \beta_{0,s-1} \\ \vdots \\ \beta_{ns} - \beta_{n-1,s-1} \end{bmatrix} = \begin{bmatrix} \varphi_{-n} \\ \vdots \\ -1 \end{bmatrix} \beta_{n,s-1} - \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \beta_{0s},$$

and so, multiplying the equation by T_{n-1}^{-1} , we get

$$(5.14) \quad \begin{bmatrix} \beta_{1s} \\ \vdots \\ \beta_{ns} \end{bmatrix} = \begin{bmatrix} \beta_{0,s-1} \\ \vdots \\ \beta_{n-1,s-1} \end{bmatrix} + r_{n-1} \cdot \beta_{n,s-1} - s_n \cdot \beta_{0s}.$$

Using this formula for $s=1, \dots, n$ we get all of the elements of T_n^{-1} .

6. Modification of the algorithm for band matrices.

Suppose that $\varphi_r = \sigma$ when $r < -q$ or $r > p$, and that φ_{-q} and φ_p are regular matrices. In this case we can give a simplified algorithm for the computation of s_m, r_m, y_m, x_m . We shall consider only the computation of s_m and r_m .

Assume that $p+q \leq m$. Let $s = \max(p, q) - 1$. Suppose that $\psi_{0,m}, \dots, \psi_{s,m}; \psi_{m-s+1,m}, \dots, \psi_{m,m}; \varrho_{0,m}, \dots, \varrho_{s,m}; \varrho_{m-s+1,m}, \dots, \varrho_{m,m}$ have been computed. From the relations (3.9), (3.11), (3.12), (3.15), (3.16), (3.17) we have immediately

$$\begin{aligned} \begin{bmatrix} \psi_{0,m+1} \\ \vdots \\ \psi_{s,m+1} \end{bmatrix} &= \begin{bmatrix} \psi_{0,m} \\ \vdots \\ \psi_{s,m} \end{bmatrix} - \begin{bmatrix} \varrho_{0,m} \\ \vdots \\ \varrho_{s,m} \end{bmatrix} \psi_{m+1,m+1} \\ \begin{bmatrix} \varrho_{1,m+1} \\ \vdots \\ \varrho_{s,m+1} \end{bmatrix} &= \begin{bmatrix} \varrho_{0,m} \\ \vdots \\ \varrho_{s-1,m} \end{bmatrix} - \begin{bmatrix} \psi_{0,m} \\ \vdots \\ \psi_{s-1,m} \end{bmatrix} \varrho_{0,m+1}, \\ \lambda_{m+1} &= \varphi_0 - \sum_{j=1}^p \varphi_j \varrho_{m-j+1,m} \\ \lambda_{m+1} &= \varphi_0 - \sum_{j=1}^q \varphi_{-j} \psi_{j-1,m} \\ \psi_{m+1,m+1} &= \lambda_{m+1}^{-1} \left(- \sum_{j=1}^p \varphi_j \psi_{m+1-j,m} \right), \\ \varrho_{0,m+1} &= \lambda_{m+1}^{-1} \left(- \sum_{j=1}^q \varphi_{-j} \varrho_{j-1,m} \right), \\ \begin{bmatrix} \psi_{m-s+1,m+1} \\ \vdots \\ \psi_{m,m+1} \end{bmatrix} &= \begin{bmatrix} \psi_{m-s+1,m} \\ \vdots \\ \psi_{m,m} \end{bmatrix} - \begin{bmatrix} \varrho_{m-s+1,m} \\ \vdots \\ \varrho_{m,m} \end{bmatrix} \psi_{m+1,m+1} \\ \begin{bmatrix} \varrho_{m+1-s,m+1} \\ \vdots \\ \varrho_{m+1,m+1} \end{bmatrix} &= \begin{bmatrix} \varrho_{m-s,m} \\ \vdots \\ \varrho_{m,m} \end{bmatrix} - \begin{bmatrix} \psi_{m-s,m} \\ \vdots \\ \psi_{m,m} \end{bmatrix} \varrho_{0,m+1}. \end{aligned}$$

For the corresponding components of x_m, y_m we can deduce similar formulas.

Now we give a simple recursion formula for the computation of the other components of s_{n-1} , r_{n-1} , x_{n-1} , y_{n-1} . For this we take

$$T_{n-1} = \begin{array}{c} \begin{array}{|c|} \hline \begin{array}{c} \varphi_0 \quad \dots \quad \varphi_{-q} \\ \vdots \\ \varphi_{p-1} \end{array} \\ \hline \end{array} \\ \begin{array}{|c|} \hline \begin{array}{c} \varphi_p \\ \vdots \\ \varphi_{q+1} \end{array} \\ \hline \end{array} \\ \begin{array}{|c|} \hline \begin{array}{c} \varphi_p \quad \dots \quad \varphi_0 \end{array} \\ \hline \end{array} \end{array}$$

and corresponding to them

$$s_{n-1}^t = (\psi_{0,n-1}, \dots, \psi_{p-1,n-1} | \psi_{p,n-1}, \dots, \psi_{n-1-q,n-1} | \psi_{n-q,n-1}, \dots, \psi_{n-1})$$

$$f_{n-1,+}^t = (\varphi_1, \dots, \varphi_p | \sigma, \dots, \sigma | \sigma, \dots, \sigma).$$

From $T_{n-1}s_{n-1} = f_{n-1,+}$ we get immediately that

$$\sum_{i=-q}^p \varphi_i \psi_{r-i,n-1} = \sigma \quad (r = p, p+1, \dots, n-1-q).$$

The same way we have

$$\sum_{i=-q}^p \varphi_i \varrho_{r-i,n-1} = \sigma \quad (r = p, p+1, \dots, n-1-q).$$

Hence we have

$$\psi_{t,n-1} = -\varphi_q^{-1} \sum_{j=-p}^{q-1} \varphi_{-j} \psi_{t-q+j,n-1}$$

$$\varrho_{t,n-1} = -\varphi_q^{-1} \sum_{j=-p}^{q-1} \varphi_{-j} \varrho_{t-q+j,n-1}$$

for $t \geq s+1$.

References

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