

A generalization of the Eulerian numbers

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Summary

In this paper we shall demonstrate that the solution of an occupancy problem leads in a natural way to a generalization of the Eulerian numbers.

1. Introduction

In a series of random trials we distribute balls at random in a set of boxes numbered $1, 2, 3, \dots$. In the n -th trial we distribute l balls ($l \geq 1$) in the first n boxes in such a way that we put each ball independently of the others in any one of the n boxes with probability $1/n$. Let $v(n)$ denote the number of empty boxes among the first n boxes at the completion of the n -th trial and define $v(0)=0$. We are concerned with the determination of the distribution of the random variable $v(n)$ for $n=0, 1, 2, \dots$.

Write

$$(1) \quad \mathbf{P}\{v(n) = k\} = P(n, k)$$

for $0 \leq k \leq n$ and define

$$(2) \quad B_r(n) = \sum_{k=r}^n \binom{k}{r} P(n, k)$$

for $0 \leq r \leq n$ as the r -th binomial moment of $v(n)$.

If we know $B_r(n)$ for $0 \leq r \leq n$, then (1) is given by the formula

$$(3) \quad P(n, k) = \sum_{r=k}^n (-1)^{r-k} \binom{r}{k} B_r(n)$$

for $0 \leq r \leq n$.

We can easily determine (2). Denote by $A_i(n)$ ($1 \leq i \leq n$) the event that box i is empty at the completion of the n -th trial. Then we have

$$(4) \quad B_r(n) = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \mathbf{P}\{A_{i_1}(n) A_{i_2}(n) \dots A_{i_r}(n)\}$$

for $1 \leq r \leq n$ and $B_0(n)=1$. Since now

$$(5) \quad \mathbf{P}\{A_{i_1}(n) A_{i_2}(n) \dots A_{i_r}(n)\} = [(i_1-1)(i_2-2) \dots (i_r-r)(n-r)!/n!]^l$$

for $1 \leq i_1 < i_2 < \dots < i_r \leq n$, therefore (1) is completely determined by (3), (4) and (5).

As an alternative approach we can use the following recurrence formula for the determination of (1):

$$(6) \quad n^l P(n, k) = \sum_{s=0}^l P(n-1, k-1+s) \binom{k+s}{k} \sum_{i=0}^s (-1)^i \binom{s}{i} (n-k-i)^l$$

for $0 \leq k \leq n$. In the sum $P(n-1, j)=0$ if $j \geq n$ or $j < 0$, and $P(0, 0)=1$.

Since in (4) the sum contains $\binom{n}{r}$ terms, the evaluation of $B_r(n)$ becomes unmanageable for large values of n . However, fortunately we can find an explicit expression for $B_r(n)$ in the form of certain generalized Stirling's numbers of the second kind.

2. Generalized Stirling's numbers

For a given positive integer l let us define $\mathfrak{S}(n, j, l)$ for $n \geq 0$ and $j \geq 0$ by the recurrence formula

$$(7) \quad \mathfrak{S}(n+1, j, l) = \mathfrak{S}(n, j-1, l) + j^l \mathfrak{S}(n, j, l)$$

where $n \geq 0$ and $j \geq 1$ and by the initial conditions $\mathfrak{S}(0, 0, l)=1$, $\mathfrak{S}(n, 0, l)=0$ for $n \geq 1$ and $\mathfrak{S}(0, j, l)=0$ for $j \geq 1$. Table 1 contains the numbers $\mathfrak{S}(n, j, l)$ for $n \leq 4$.

TABLE 1 for $\mathfrak{S}(n, j, l)$.

$n \backslash j$	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	2^l+1	1	0
4	0	1	4^l+2^l+1	3^l+2^l+1	1

We have

$$(8) \quad \mathfrak{S}(n, j, l) = \sum_{1 \leq r_1 \leq r_2 \leq \dots \leq r_{n-j} \leq j} (r_1 r_2 \dots r_{n-j})^l$$

for $1 \leq j \leq n$ and $\mathfrak{S}(n, j, l)=0$ for $0 < n < j$. For if $\mathfrak{S}(n, j, l)$ is defined by (8), then (7) is satisfied for $n \geq 1$ and $\mathfrak{S}(n, 1, l)=1$ for $n \geq 1$.

From (7) we obtain the generating function

$$(9) \quad \sum_{n=j}^{\infty} \mathfrak{S}(n, j, l) x^n = \prod_{r=1}^j \left(\frac{x}{1-r^l x} \right)$$

for $j \geq 1$ and $|x| < j^{-l}$ and by partial fraction expansion we get the explicit expression

$$(10) \quad \mathfrak{S}(n, j, l) = \sum_{i=0}^j (-1)^{j-i} A_i(l) B_{ij}(l) i^{nl}$$

for $0 \leq j \leq n$ where

$$(11) \quad A_i(l) = \prod_{0 \leq r < i} (i^l - r^l)^{-1}$$

for $i \geq 0$ ($A_0(l)=1$) and

$$(12) \quad B_{ij}(l) = \prod_{i < r \leq j} (r^l - i^l)^{-1}$$

for $0 \leq i \leq j$ ($B_{jj}(l)=1$).

If $l=1$, then $\mathfrak{S}(n, j, l) = \mathfrak{S}(n, j)$ where $\mathfrak{S}(n, j)$ ($0 \leq j \leq n$) are Stirling's numbers of the second kind introduced in 1730 by J. STIRLING [59]. See also Ch. JORDAN [34], [35]. If $l=1$ in (10), we get

$$(13) \quad \mathfrak{S}(n, j) = \frac{1}{j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^n$$

for $0 \leq j \leq n$.

If $l=2$, then $\mathfrak{S}(n, j, l) = T(2n, 2n-j)$ where $T(n, k)$ ($0 \leq k \leq n$) are central factorial numbers. See J. LOHNE [41], L. CARLITZ and J. RIORDAN [20], and J. RIORDAN [52, p. 217].

For $l \geq 1$ the numbers $\mathfrak{S}(n, j, l)$ are particular cases of more general numbers introduced in 1853 by A. CAYLEY [23]. See also O. J. MUNCH [47] and L. COMTET [24].

3. The probability $P(n, k)$

By (5) and (8) we can express (4) in the following explicit form

$$(14) \quad B_r(n) = \mathfrak{S}(n, n-r, l) [(n-r)!]^l / (n!)^l$$

for $0 \leq r \leq n$ and thus by (3) we get

$$(15) \quad P(n, k) = A(n, k, l) / (n!)^l$$

for $0 \leq k \leq n$ where

$$(16) \quad A(n, k, l) = \sum_{r=k}^n (-1)^{r-k} \binom{r}{k} \mathfrak{S}(n, n-r, l) [(n-r)!]^l$$

for $n \geq 0$, $k \geq 0$ and $l \geq 1$. Clearly $A(n, k, l) = 0$ if $k \geq n \geq 1$.

Let us define the polynomials

$$(17) \quad A_n(x, l) = \sum_{k=0}^n A(n, k, l) x^k$$

for $n \geq 0$. In particular we have $A_0(x, l) = 1$, $A_1(x, l) = 1$, $A_2(x, l) = 2^l - 1 + x$, $A_3(x, l) = 6^l - 4^l - 2^l + 1 + (4^l + 2^l - 2)x + x^2$, $A_4(x, l) = (24)^l - (18)^l - (12)^l + 8^l - 6^l + 4^l + 2^l - 1 + [(18)^l + (12)^l - 2(8)^l + 6^l - 2(4)^l - 2(2)^l + 3]x + (8^l + 4^l + 2^l - 3)x^2 + x^3$.

In the particular case where $l=1$, let us write $A(n, k, l) = A(n, k)$ and $A_n(x, l) = A_n(x)$. By (10) and (16) we have

$$(18) \quad A(n, k) = A(n, n-1-k) = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n$$

for $0 \leq k \leq n-1$ and $A(n, k) = 0$ for $k \geq n \geq 1$. Table 2 contains the numbers (18) for $n \leq 6$.

TABLE 2 for $A(n, k)$

$n \backslash k$	0	1	2	3	4	5
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	4	1	0	0	0
4	1	11	11	1	0	0
5	1	26	66	26	1	0
6	1	57	302	302	57	1

The numbers $A(n, k)$ defined by (18) are called Eulerian numbers. It seems L. EULER [30] was the first who studied these numbers in 1736. The numbers $A(n, k, l)$ defined by (16) are some natural generalizations of the numbers $A(n, k)$.

By the results of this paper the numbers $A(n, k, l)$ have a simple combinatorial interpretation. Let us consider a triangular board of size n in which the i -th row contains i cells ($i=1, 2, \dots, n$). (See Figure 1)

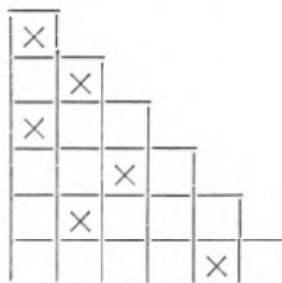


Figure 1

Let us put a mark in a cell in each row of the triangle. We can interpret $A(n, k, l)$ as the number of ways in which we can superimpose l marked triangles such that there are exactly k empty columns.

4. Historical notes

In 1736 L. EULER [30], [31 pp. 373—374] defined the polynomials $A_n(x) = A_n(x, 1)$ as the coefficients in his new summation formula. He determined the explicit form of $A_n(x)$ and proved that

$$(19) \quad \frac{x-1}{x-e^{u(x-1)}} = \sum_{n=0}^{\infty} A_n(x) u^n / n!$$

for $x \neq 1$ and sufficiently small u .

In 1812 P. S. LAPLACE [40 pp. 256—257] demonstrated that if $\xi_1, \xi_2, \dots, \xi_n$ are mutually independent random variables each having a uniform distribution

over the interval $(0, 1)$, then

$$(20) \quad \mathbf{P}\{\xi_1 + \dots + \xi_n \leq x\} = \frac{1}{n!} \sum_{j=0}^{\lfloor x \rfloor} (-1)^j \binom{n}{j} (x-j)^n$$

for $0 \leq x \leq n$. On the other hand if in (15) $l=1$, we have

$$(21) \quad \mathbf{P}\{v(n) \leq k\} = \mathbf{P}\{v(n) \geq n-1-k\} = \frac{1}{n!} \sum_{j=0}^{k+1} (-1)^j \binom{n}{j} (k+1-j)^n$$

for $k=0, 1, \dots, n-1$. A comparison of (20) and (21) shows that

$$(22) \quad \mathbf{P}\{v(n) < k\} = \mathbf{P}\{\xi_1 + \dots + \xi_n \leq k\}$$

for $k=1, 2, \dots, n$. For a simple proof of (20) see reference [60]. By (22) it follows that $v(n)$ has the same distribution as $[\xi_1 + \dots + \xi_n]$. This representation of $v(n)$ makes it possible to determine easily the asymptotic behavior of $A(n, k)$ as $n \rightarrow \infty$ and $k \rightarrow \infty$. For example by the central limit theorem it follows that

$$(23) \quad \lim_{n \rightarrow \infty} \mathbf{P}\left\{\frac{2v(n)-n}{\sqrt{n/3}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

for any x .

In analysis, number theory, combinatorial analysis and probability theory, several results are connected with Eulerian numbers.

In studying Fermat's last problem in 1857 E. E. KUMMER [39] introduced the polynomials

$$(24) \quad P_n(x, y) = (x+y)^n \left(\frac{d^n \log(x+e^u y)}{du^n} \right)_{u=0}$$

for $n \geq 1$. It can easily be seen that $P_n(xy, y) = y^n P_n(x, 1)$ and $P_n(x, 1) = x A_{n-1}(-x)$ for $n \geq 1$. The numbers

$$(25) \quad B_n = n P_n(1, 1) / (2^n - 1)$$

defined for $n \geq 2$ are Bernoulli numbers.

In 1883 J. WÖRPTZKY [68] demonstrated that the Eulerian numbers (18) satisfy the recurrence formula

$$(26) \quad A(n+1, k) = (k+1) A(n, k) + (n+1-k) A(n, k-1)$$

for $n \geq 1$ and $k \geq 1$ where $A(n, 0) = 1$ for $n \geq 1$ and $A(1, k) = 0$ for $k \geq 1$. Furthermore, he showed that

$$(27) \quad x^n = \sum_{k=0}^{n-1} A(n, k) \binom{x+k}{n}$$

for $n \geq 1$. See also H. F. SCHERK [55], E. UNFERDINGER [64], G. FROBENIUS [33], L. KRONECKER [38], L. SAALSCHÜTZ [54], D. MIRIMANOFF [45], and H. S. VANDIVER [65].

In the last quarter of the nineteenth century the works of J. BIENAYMÉ [8] and D. ANDRÉ [1], [2], [3], [4], [5] focused attention on the study of random permutations. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be a random permutation of $(1, 2, \dots, n)$ and suppose that every possible permutation has the same probability. In 1879 D. ANDRÉ [1] proved that

$$(28) \quad P\{\alpha_1 < \alpha_2, \alpha_2 > \alpha_3, \alpha_3 < \alpha_4, \dots\} = A(n)/n!$$

for $n \geq 2$ where

$$(29) \quad \sum_{n=0}^{\infty} A(n) \frac{x^n}{n!} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sec x + \tan x.$$

In 1908 P. A. MACMAHON [42] by solving a problem of Simon Newcomb concerning a card game proved that the probability that a deck of n cards numbered $1, 2, \dots, n$ may be dealt into $k+1$ piles, if cards are placed in one pack as long as they are in descending order of magnitude, is given by

$$(30) \quad P\{\alpha_i < \alpha_{i+1} \text{ for exactly } k \text{ subscripts } i = 1, 2, \dots, n-1\} A = (n, k)/n!$$

for $k=0, 1, \dots, n-1$. The same result has been obtained in various forms by L. SCHRUTKA [56], G. H. MOORE and W. A. WALLIS [46], and J. RIORDAN [50]. See also I. KAPLANSKY and J. RIORDAN [36], R. SPRAGUE [58], L. CARLITZ and J. RIORDAN [19], D. E. BARTON and F. N. DAVID [6], and D. E. BARTON and C. L. MALLOWS [7]. Various generalizations of the problem of Simon Newcomb have been given by P. A. MACMAHON [42], G. KREWERAS [37], L. CARLITZ [14], [15], and J. F. DILLON and D. P. ROSELLE [26].

By the solution of the Smith College diploma problem of R. MAUER [44] we have the following result

$$(31) \quad P\{\alpha_i < i \text{ for exactly } k \text{ subscripts } i = 1, 2, \dots, n-1\} = A(n, k)/n!$$

for $k=0, 1, \dots, n-1$. See D. WEST [66] and D. FOATA and M.-P. SCHÜTZENBERGER [32].

In 1974 D. DUMONT [27] proved that there are exactly $A(n, k)$ arrangements $(u_1, u_2, \dots, u_n)$ such that $i \leq u_i \leq n$ for $i=1, 2, \dots, n$ and $(u_1, u_2, \dots, u_n)$ contains $k+1$ distinct integers. This interpretation of $A(n, k)$ is in agreement with our interpretation of $A(n, k, l)$ for $l=1$.

Worpitzky's formula (27) has been rediscovered several times. See e.g. P. S. DWYER [28], [29] and P. A. PIZA [48]. Formula (27) has been generalized by E. B. SHANKS [57] and L. CARLITZ [9].

The asymptotic result (23) has been proved by J. WOLFOWITZ [67], H. B. MANN [43], L. CARLITZ, D. C. KURTZ, R. SCOVILLE and O. P. STACKELBERG [18], and S. TANNY [61].

For the general theory of Eulerian numbers we refer to L. CARLITZ [11], [17], J. RIORDAN [51], and D. FOATA and M.-P. SCHÜTZENBERGER [32]. See also L. TOSCANO [62], [63], L. CARLITZ [13], L. CARLITZ, D. P. ROSELLE and R. A. SCOVILLE [21], D. P. ROSELLE [53] and F. POUSSIN [49].

By generalizing (27) L. CARLITZ [10] introduced the so-called q -Eulerian numbers. Eulerian numbers of higher order have been defined by L. CARLITZ [12]. See

also J. F. DILLON and D. P. ROSELLE [25]. In 1973 L. CARLITZ [16] introduced generalized Eulerian numbers by expanding the k -th power of (19) into a Taylor series. See also L. CARLITZ [17], and L. CARLITZ and R. SCOVILLE [22].

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