

A theorem on error bounds in $L_2(0, 2\pi)$ space

By P. GOYALIYA (Calgary, Alberta)

1. JACKSON, D. [2] obtained the error bounds of functions having modulus of continuity, by means of trigonometric polynomials in $L(0, 2\pi)$ space. He also discussed the same aspect for functions having one or more derivatives. The problem of error bounds in $L_2(0, 2\pi)$ space has been studied by N. I. CERNYH [1]. In this note we have obtained the error bounds of functions having one or more derivatives in $L_2(0, 2\pi)$ space.

2. Let $L_2(0, 2\pi)$ be the space of measurable functions $f(x)$ for which $\int_0^{2\pi} |f(x)|^2 dx < \infty$. Let

$$(2.1) \quad \|f(x)\|_{L_2} = \left\{ \frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 dx \right\}^{1/2},$$

$$(2.2) \quad w(\delta, f)_{L_2} = \sup_{|t| \leq \delta} \|f(x+t) - f(x)\|_{L_2}$$

and

$$(2.3) \quad E_n(f)_{L_2} = \|f(x) - s_n(x; f)\|_{L_2}$$

where $s_n(x, f)$ are the partial sums of order n of the Fourier series of $f(x)$.

Throughout the note, M , without suffixes denotes a positive constant, not necessarily the same at each occurrence.

3. On Jackson's inequality in $L_2(0, 2\pi)$ space N. I. CERNYH [1] proved the following Theorem.

Theorem A. Let $f(x) \in L_2(0, 2\pi)$, $f(x) \neq \text{constant}$. For $n=0, 1, 2, \dots$ we have

$$(3.1) \quad E_n(f)_{L_2} \leq \frac{1}{\sqrt{2}} w\left(\frac{\pi}{n+1}, f\right)_{L_2}.$$

We generalize the above Theorem in the following form.

Theorem 1. $f^{(m)}(x) \in L_2(0, 2\pi)$ $m \geq 0$, $f^{(m)}(x) \neq \text{constant}$. For $n=0, 1, 2, \dots$, we have

$$(3.2) \quad E_n(f^{(m)})_{L_2} \leq \frac{1}{\sqrt{2}} w\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2},$$

and

$$(3.3) \quad E_n(f)_{L_2} \leq \frac{M}{n^m} E_n(f^{(m)})_{L_2}.$$

It is interesting to note that for $m=0$, Theorem 1 reduces to Theorem A. 4. We require the following lemma for the proof of our Theorem.

Lemma 1. *Let $N \geq 1$ be an integer, let $l_k (k \geq N)$ be real no's and $0 < \sum l_k^2 < \infty$, then the function*

$$(4.1) \quad F_N(t) = \sum_{k=N}^{\infty} l_k^2 \cos kt$$

takes both positive and negative values on $(0, \pi/N)$.

PROOF OF LEMMA. Put

$$\varphi_N(t) = \begin{cases} -\sin Nt & \text{for } 0 \leq t \leq \frac{\pi}{N}, \\ 0 & \text{for } \frac{\pi}{N} \leq t \leq \pi. \end{cases}$$

Expanding this function in a Fourier series of the form

$$\varphi_N(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos kt,$$

we easily see that $A_N = 0$ and

$$A_k = \frac{4N}{\pi} \frac{1}{k^2 - N^2} \cos^2 \frac{k\pi}{2N} \geq 0 \quad (k > N).$$

From this we obtain by Parseval's equation

$$\frac{1}{\pi} \int_0^{\pi} \varphi_N(t) F_N(t) dt = \sum_{k=N}^{\infty} l_k^2 A_k \geq 0.$$

If we suppose that $F_N(t) \geq 0$ for $0 < t < \frac{\pi}{N}$ then, by the definition of $\varphi_N(t)$ and the fact that $F_N(t) > 0$ in a neighbourhood of $t=0$, we obtain the inequality

$$\int_0^{\pi} \varphi_N(t) F_N(t) dt < 0$$

which contradicts the preceding one. This establishes the lemma.

5. PROOF OF THEOREM 1.

Let
$$f(x) \sim \sum_1^{\infty} (a_k \cos kx + b_k \sin kx)$$
 then

$$f^{(m)}(x) \sim \sum_1^{\infty} k^m (a_k \cos kx + b_k \sin kx).$$

Now

$$(5.1) \quad \begin{aligned} E_n^2(f^{(m)}) &= \|f^{(m)}(x) - S_n(x, f^{(m)})\|^2 = \\ &= \left\| \sum_{k=1}^{\infty} k^m (a_k \cos kx + b_k \sin kx) \right\|^2 = \sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2), \end{aligned}$$

by Parseval's inequality.

Also

$$(5.2) \quad \begin{aligned} &\|f^{(m)}(x+t) - f^{(m)}(x)\|_{L_2}^2 = \\ &= \left\| \sum_{k=1}^{\infty} k^m (a_k \cos k(x+t) + b_k \sin k(x+t)) - \sum_{k=1}^{\infty} k^m (a_k \cos kx + b_k \sin kx) \right\|_{L_2}^2 = \\ &= 2 \sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) (1 - \cos kt) \end{aligned}$$

by Parseval's inequality.

Since $1 - \cos kt \geq 0$, we have, from (5.2)

$$\begin{aligned} w^2(\pi/n+1, f^{(m)})_{L_2} &= 2 \sup_{|t| < \frac{\pi}{n+1}} \sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) (1 - \cos kt) \geq \\ &\geq 2 \sup_{|t| < \frac{\pi}{n+1}} \sum_{k=n+1}^{\infty} k^{2m} (a_k^2 + b_k^2) (1 - \cos kt) \geq \\ &\geq 2 \sum_{k=n+1}^{\infty} k^{2m} (a_k^2 + b_k^2) - 2 \inf_{|t| < \frac{\pi}{n+1}} \sum_{k=n+1}^{\infty} k^{2m} (a_k^2 + b_k^2) \cos kt \geq \\ &\geq 2 \sum_{k=n+1}^{\infty} k^{2m} (a_k^2 + b_k^2), \end{aligned}$$

by Lemma 1 ($\inf \sum k^{2m} (a_k^2 + b_k^2) \cos kt \geq 0$).

Now we have

$$2E_n^2(f^{(m)})_{L_2} = 2 \sum_{k=n+1}^{\infty} k^{2m} (a_k^2 + b_k^2) \geq w^2\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2}$$

(by (5.1)), or

$$E_n^2(f^{(m)})_{L_2} \geq \frac{1}{\sqrt{2}} w^2\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2}.$$

Now, to prove (3.3) we have

$$E_n^2(f) = \|f(x) - S_n(x, f)\|^2 = \left\| \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right\|^2 = \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

by Parseval's inequality

$$\leq \frac{M}{n^{2m}} \sum_{k=1}^{\infty} k^{2m} (a_k^2 + b_k^2) = \frac{M}{n^{2m}} E_n^2(f^{(m)}) \quad (M \text{ is a constant})$$

i.e.

$$E_n(f)_{L_2} \leq \frac{M}{n^m} E_n(f^{(m)})_{L_2}.$$

This proves Theorem 1.

6. It may be interesting to know the answers to the following questions:

- (i) Can Theorem 1 be extended to $(c, 1)$ summability, or Matrix summability?
- (ii) Can the result be extended to some other series, viz. Legendre series, ultraspherical series, Bessel's series, etc.?

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References

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALGARY
CALGARY, ALBERTA, CANADA

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