

Remarks on the limit inferior of a filtered set-family

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It is well-known (see KURATOWSKI [6], p. 242) that in each metric space E the limit inferior $\liminf_{n \rightarrow \infty} A_n$ of each sequence (A_n) with $A \subseteq E$ is the set of all $x \in E$ such that there exist an $n_0(x) \in \mathbb{N}$ and a sequence $(a_n(x))$ in E with $a_n(x) \in A_n$ for all $n \geq n_0(x)$ and $x = \lim_{n \rightarrow \infty} a_n(x)$. In this paper, we give (in Proposition 3) a modified generalization of this statement to a (general) topological space E and the limit inferior $\liminf(f, I, \alpha)$ of a filtered family (f, I, α) with $f(i) \subseteq E$ for all $i \in I$.

For the remainder, let (E, τ) be a topological space, $\mathfrak{B}_\tau$ the neighborhood operator, Lim_τ the limit operator, $\liminf_\tau$ the limit inferior induced by the topology τ . For abbreviation, we write just $\mathfrak{B}$, Lim , $\liminf$ instead of $\mathfrak{B}_\tau$, Lim_τ , $\liminf_\tau$ if no confusion can arise.

1. Terminology. In every respect we shall use the same terminology as used or introduced in [4]. For nonempty sets I and K and filters α and β on I and K , respectively, $\alpha \otimes \beta$ denotes the ordinal product of α and β (which is a filter on $I \times K$) (see [3], p. 330 and p. 336, Satz 23). Given a filter α on a set I and $A \in \alpha$, α_A denotes the trace of α in the set A ; furthermore, if, for each $i \in I$, a statement form $H(i)$ containing i as a free variable is given, we say that $H(i)$ holds "for α -almost all $i \in I$ " if and only if, for some $A \in \alpha$, $H(i)$ holds for all $i \in A$. Given a mapping f and a set $B \subseteq \mathcal{D}f$, we denote the restriction of f to B by f_B . Recall that, for each set M , ΦM denotes the class of all filtered families in M . For each directed set $(D, \equiv)$, i.e. for each set D with a reflexive and transitive relation $\equiv$ such that each finite subset of D has an upper bound w.r. to $\equiv$, we denote by $\mathfrak{F}D$ the "filter of perfinality" on D , which is defined to be the filter on D generated by the set $\{\{z | y \equiv z \in D\} | y \in D\}$. Especially, for each $x \in E$, $(\mathfrak{B}x, \supseteq)$ is a directed set, and so $\mathfrak{F}(\mathfrak{B}x)$ is the filter of perfinality on $\mathfrak{B}x$.

2. Connection between $\liminf$ and Lim . The mapping $\liminf$ is an extension of the mapping Lim in the sense of

Proposition 1. Let κ denote the mapping $x \mapsto \{x\}$ on E into $\mathfrak{P}E$. Then

$$\text{Lim}(f, I, \alpha) = \liminf(\kappa \circ f, I, \alpha)$$

holds for all $(f, I, \alpha) \in \Phi E$.

PROOF. Use of the definitions only. $\square$

While Proposition 1 remains true in general finitely additive quasitopological spaces (see Section 4), this is not the case for the next proposition:

Proposition 1'. *In the notation of Proposition 1, one has:*

$$\text{Lim}(f, I, \alpha) = \liminf_{n \rightarrow \infty} (\tau \circ \kappa \circ f, I, \alpha)$$

holds for all $(f, I, \alpha) \in \Phi E$.

PROOF. Use that, for each $x \in E$, $\mathfrak{B}_x$ is generated by the class of all open neighborhoods of x ; furthermore that $f(i) \in \tau \{f(i)\}$ for all $i \in I$. $\square$

One half of the statement on $\liminf_{n \rightarrow \infty} A_n$ in the introduction is still true in general topological spaces:

Proposition 2. *For all $x \in E$ and all $(f, I, \alpha) \in \Phi(\mathfrak{P}E)$, the following statement form (a) implies (b):*

- (a) *There are an $A \in \alpha$ and a $g \in \prod_{i \in A} f(i)$ such that $x \in \text{Lim}(g, A, \alpha_A)$.*
- (b) $x \in \liminf(f, I, \alpha)$.

PROOF. Use of the definitions of Lim and $\liminf$ in terms of $\mathfrak{B}$. $\square$

The full generalization of the introductory remark on $\liminf_{n \rightarrow \infty} A_n$ is given by:

Proposition 3. *For all $x \in E$ and all $(f, I, \alpha) \in \Phi(\mathfrak{P}E)$, the following statement forms (a) through (d) are equivalent:*

- (a) $x \in \liminf(f, I, \alpha)$;
- (b) *There exists a mapping g on $(\mathfrak{B}_x) \times I$ into E with the following property: $g(V, i) \in f(i)$ for $((\mathfrak{F}(\mathfrak{B}_x)) \otimes \alpha)$ -almost all $(V, i) \in (\mathfrak{B}_x) \times I$ and $x \in \text{Lim}(g, (\mathfrak{B}_x) \times I, (\mathfrak{F}(\mathfrak{B}_x)) \otimes \alpha)$.*
- (c) *is obtained from (b) by replacing the quantifier "for $((\mathfrak{F}(\mathfrak{B}_x)) \otimes \alpha)$ -almost all" by "for $\{\mathfrak{B}_x\} \otimes \alpha$ -almost all".*
- (d) *There are a $C \in (\mathfrak{F}(\mathfrak{B}_x)) \otimes \alpha$ and a $g \in \prod_{(V, i) \in C} f(i)$ such that*

$$x \in \text{Lim}(g, C, ((\mathfrak{F}(\mathfrak{B}_x)) \otimes \alpha)_C).$$

PROOF. First recall that

$$(1) \quad \text{Lim}(h, K, x) = \text{Lim}(h_X, X, x_X)$$

holds for each $(h, K, x) \in \Phi E$ and each $X \in x$. Let $x \in E$ and $(f, I, \alpha) \in \Phi(\mathfrak{P}E)$. For abbreviation we put $\mathfrak{B}_x = D$. Now we subdivide the proof.

1. By (1), (b) and (d) are equivalent.

2. Since $D \in \mathfrak{F}D$, (c) implies (b).

3. We show that (a) implies (c). Suppose (a). Then, there is, for each $V \in D$, a set $A_V \in \alpha$ such that

$$(2) \quad V \cap f(i) \neq \emptyset \quad \text{for all } i \in A_V.$$

Let $(A_V)_{V \in D}$ be a family of such sets A_V (use of the axiom of choice) and $\mathfrak{B}$ the direct sum $\sum_{V \in D} A_V$ of the sets A_V (see [3], p. 325). Then (by (2)) there exists a $\chi \in \prod_{(V, i) \in \mathfrak{B}} V \cap f(i)$ (use of the axiom of choice). Define a mapping g on $D \times I$ into E by letting

$$g(V, i) = \begin{cases} \chi(V, i) & \text{for all } (V, i) \in \mathfrak{B}, \\ x & \text{for all } (V, i) \in (D \times I) \setminus \mathfrak{B}. \end{cases}$$

Since $\mathfrak{B} \in \{D\} \otimes \alpha$, we have

$$g(V, i) \in f(i) \quad \text{for} \quad \{D\} \otimes \alpha\text{-almost all} \quad (V, i) \in D \times I.$$

Let $W \in D$. Then, by the definition of g ,

$$g(V, i) \in V \cap f(i) \subseteq W \cap f(i) \subseteq W$$

holds for all (V, i) with $V \subseteq W$, $V \in D$ and $i \in A_V$, thus for $(\mathfrak{F}D) \otimes \alpha$ -almost all $(V, i) \in D \times I$. By the choice of W , we obtain $x \in \text{Lim}(g, D \times I, (\mathfrak{F}D) \otimes \alpha)$. Therefore, (c) holds.

4. We show that (b) implies (a). Suppose (b). Let $V \in D$. Then, with the mapping g taken from (b), $g(W, i) \in V \cap f(i)$ holds for $(\mathfrak{F}D) \otimes \alpha$ -almost all $(W, i) \in D \times I$ (consider that $(\mathfrak{F}D) \otimes \alpha$ is a filter). Therefore, by the definition of $\mathfrak{F}D$ and of $\otimes$ (use of [3], p. 330, Satz 13), there exists a set $U \in D$ and, for each $W \in D$ with $W \subseteq U$, a set $A_W \in \alpha$ such that

$$g(W, i) \in V \cap f(i) \quad \text{for all} \quad i \in A_W.$$

Especially, we have $V \cap f(i) \neq \emptyset$ for all $i \in A_U$. By the choice of V , we obtain (a). $\square$

3. Applications of the preceding propositions. It would be desirable to regain "nice" properties of certain subspaces of $(\mathfrak{P}E, \mathfrak{P}\tau)$ (the power of (E, τ)) from corresponding or related properties of (E, τ) , or conversely, by means of the preceding propositions. A simple step in the desired direction can be seen in the next proposition.

Proposition 4. *Let $\mathfrak{M} \subseteq (\mathfrak{P}E) \setminus \{\emptyset\}$ and assume $\{x\} \in \mathfrak{M}$ for all $x \in E$. Then, the subspace $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ of $(\mathfrak{P}E, \mathfrak{P}\tau)$ is compact if and only if (E, τ) is compact.*

PROOF. 1. Assume $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ to be compact. Let $(f, I, \alpha) \in \Phi E$ with an ultrafilter α . In the terminology within Proposition 1, one has $\mathfrak{M} \cap (\mathfrak{P} \liminf (\kappa \circ f, I, \alpha)) \neq \emptyset$, since $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ is compact. Therefore, there is an $M \in \mathfrak{M}$ with $M \subseteq \liminf (\kappa \circ f, I, \alpha) = \text{Lim}(f, I, \alpha)$ (use of Proposition 1), thus (since $M \neq \emptyset$) $\text{Lim}(f, I, \alpha) \neq \emptyset$. Therefore, by the choice of (f, I, α) , (E, τ) is compact.

2. Assume that (E, τ) is compact. Let $(f, I, \alpha) \in \Phi \mathfrak{M}$ with an ultrafilter α . Since $\emptyset \notin \mathfrak{M}$, there is a $g \in \prod_{i \in I} f(i)$ (axiom of choice); then, $(g, I, \alpha) \in \Phi E$ and $\text{Lim}(g, I, \alpha)$ is nonempty, since (E, τ) is compact. In view of Proposition 2, $\liminf (f, I, \alpha)$ is nonempty; choose $x \in \liminf (f, I, \alpha)$. Then $\{x\} \in \mathfrak{M} \cap (\mathfrak{P} \liminf (f, I, \alpha))$. By the choice of (f, I, α) , $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ is proved to be compact. $\square$

While the proof of Proposition 4 can be carried over, word by word, to finitely additive quasitopological spaces (see Section 4), this is not the case for the proof of the next proposition:

Proposition 4'. *Let $\mathfrak{M} \subseteq (\mathfrak{P}E) \setminus \{\emptyset\}$ and assume $\tau\{x\} \in \mathfrak{M}$ for all $x \in E$. Then, the subspace $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ of $(\mathfrak{P}E, \mathfrak{P}\tau)$ is compact if and only if (E, τ) is compact.*

PROOF. 1. In Part 1 of the preceding proof, replace $(\kappa \circ f, I, \alpha)$ by $(\tau \circ \kappa \circ f, I, \alpha)$ and the reference to Proposition 1 by that to Proposition 1'.

2. In Part 2 of the preceding proof, replace $\{x\}$ by $\tau\{x\}$. Use that $\liminf (f, I, \alpha)$ is closed there. $\square$

Remark. Proposition 4 contains the special case $\mathfrak{M} = \mathfrak{P}E \setminus \{\emptyset\}$. In this case, $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ coincides (see [5]) with the hyperspace of lower semicontinuity of (E, τ) (see MICHAEL [7], p. 179, Definition 9.1 (“ $\mathfrak{P}E \setminus \{\emptyset\}$ with the lower finite topology”), and (for “closure spaces”) ČECH [1], p. 623, Definition 34 A.1). Proposition 4' contains the special case $\mathfrak{M} = 2^E$ (= set of all nonempty closed subsets of (E, τ)). In this case, $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ coincides (see [5] or FLACHSMEYER [2], p. 326, 2.1, or POPPE [8]) with Michael's space 2^E endowed with the “lower finite topology” ([7], loc. cit.). While this special case of Proposition 4' occurs in the literature (with a different proof, using Alexander's Lemma (see FLACHSMEYER [2], p. 327, 2.4)), the author could not find a reference concerning the mentioned special case of Proposition 4.

In the paper [4], we have considered topological spaces (E, τ) , (F, σ) , (G, λ) , relations $R \subseteq E \times F$, $S \subseteq F \times G$ with $\mathcal{R}R \subseteq \mathcal{D}S$, and — respectively — the canonical mapping $\hat{R}, \hat{S}, \widehat{S \circ R}$ induced by $R, S, S \circ R$, which is a mapping *from* (in general not *on*!) (E, τ) , (F, σ) , (E, τ) into the power $(\mathfrak{P}F, \mathfrak{P}\sigma)$, $(\mathfrak{P}G, \mathfrak{P}\lambda)$, $(\mathfrak{P}G, \mathfrak{P}\lambda)$ of (F, σ) , (G, λ) , (G, λ) , as we do now for the following. We recall the validity of the logical diagram

$$\begin{array}{ccc} R \text{ and } S \text{ continuous} & \Rightarrow & \hat{R} \text{ and } \hat{S} \text{ continuous} \\ \Downarrow & & \Downarrow \\ S \circ R & \text{continuous} \Rightarrow & \widehat{S \circ R} \text{ continuous} \end{array}$$

and use Proposition 3 to reprove the right-hand arrow, more precisely, to reprove the next proposition (cf. ČECH [1], p. 631, Theorem 34 B. 14, and [5]; furthermore, see [4], p. 41, Proposition 7):

Proposition 5. *If $\hat{R}$ is $(\tau, \mathfrak{P}\sigma)$ -continuous and $\hat{S}$ is $(\sigma, \mathfrak{P}\lambda)$ -continuous, then $\widehat{S \circ R}$ is $(\tau, \mathfrak{P}\lambda)$ -continuous.*

PROOF. Let $(f, I, \alpha) \in \Phi(\mathcal{D}R)$ and $x \in (\mathcal{D}R) \cap \text{Lim}_\tau(f, I, \alpha)$. Let $y \in (\widehat{S \circ R})(x)$ and $U \in \mathfrak{B}_\lambda y$. Then, there exists a $z \in F$ with $(x, z) \in R$ and $(z, y) \in S$. Since $\hat{R}$ is $(\tau, \mathfrak{P}\sigma)$ -continuous, we have $\hat{R}(x) \subseteq \liminf_\sigma(\hat{R} \circ f, I, \alpha)$. Therefore, since $z \in \hat{R}(x)$, $z \in \liminf_\sigma(\hat{R} \circ f, I, \alpha)$. For abbreviation, we set $(\mathfrak{F}(\mathfrak{B}_\sigma z)) \otimes \alpha = \mathfrak{c}$. By Proposition 3, there is a set $C \in \mathfrak{c}$ and a mapping $g \in \mathbf{P}_{(V, i) \in C}(\hat{R} \circ f)(i)$ such that

$$(3) \quad z \in \text{Lim}_\sigma(g, C, \mathfrak{c}_C).$$

Since $\mathcal{R}R \subseteq \mathcal{D}S$ (by supposition), one has $\mathcal{D}(\hat{S} \circ g) = C$ (in view of the choice of g). Because $\hat{S}$ is $(\sigma, \mathfrak{P}\lambda)$ -continuous, one obtains by (3)

$$(4) \quad \hat{S}(z) \subseteq \liminf_\lambda(\hat{S} \circ g, C, \mathfrak{c}_C).$$

Since $y \in \hat{S}(z)$ and $U \in \mathfrak{B}_\lambda y$, there exists, by (4), a set $D \in \mathfrak{c}_C$ such that

$$(5) \quad U \cap ((\hat{S} \circ g)(V, i)) \neq \emptyset \quad \text{for all } (V, i) \in D.$$

Because $\mathfrak{c}_C \subseteq \mathfrak{c}$, there are (by “Satz 13” in [3], p. 330) a set $W \in \mathfrak{B}_\sigma z$ and a family $(A_V)_{V \in \mathfrak{B}}$, where $\mathfrak{B}$ denotes the set of all $V \in \mathfrak{B}_\sigma z$ with $V \subseteq W$, such that $A_V \in \alpha$ for all $V \in \mathfrak{B}$ and $\bigcup_{V \in \mathfrak{B}} A_V \subseteq D$. We set $A_W = J$ (for abbreviation) and choose a mapping $\chi \in \mathbf{P}_{i \in J}(U \cap ((\hat{S} \circ g)(W, i)))$ (use of (5) and the axiom of choice).

Let $i \in J$. Then, by the choice of g , $(f(i), g(W, i)) \in R$ and, by the choice of χ , $(g(W, i), \chi(i)) \in S$, thus $(f(i), \chi(i)) \in S \circ R$, therefore $\chi(i) \in (\widehat{S \circ R})(f(i))$. On the other hand, we have $\chi(i) \in U$, thus, for all $i \in J$ (therefore, for α -almost all $i \in I$), $U \cap ((\widehat{S \circ R})(f(i))) \neq \emptyset$. In view of the choice of U , we obtain

$$y \in \liminf_{\lambda} ((\widehat{S \circ R}) \circ f, I, \alpha).$$

We have proved that $\widehat{S \circ R}$ is $(\tau, \mathfrak{B}\lambda)$ -continuous. $\square$

4. Generalization. The Propositions 1, 2, 3, 4, 5 and their proofs remain valid if the topological spaces (E, τ) , (F, σ) , (G, λ) are replaced by general finitely additive quasitopological spaces (terminology: [5]) except for the following change within Proposition 3: Within (b), replace "There exists" by the words " $\mathfrak{B}x$ is a filter and there exists". Within (d), replace "There are" by the words " $\mathfrak{B}x$ is a filter and there are". — A quasitopological space (E, τ) is called to be compact if and only if $(\text{Lim}_{\tau} \alpha \neq \emptyset$ for all ultrafilters α in E .

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