

On n -distributive system of elements of a modular lattice

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1. Introduction

A modular lattice is called n -distributive [2] if it satisfies the identity

$$D_n: x \wedge \bigvee_{i=0}^n y_i = \bigvee_{j=0}^n [x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i].$$

In [2] we proved the following

Theorem A. *Let M be a modular lattice. Then the following conditions are equivalent to each other and to their duals:*

- (i) M satisfies D_n .
- (ii) M satisfies the identity

$$M_n: \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i.$$

(For the “ n -tributive” identity M_n see also BERGMAN [1].)

It is well-known that in the classical case $n=1$ there is a stronger, local version of the theorem:

Theorem B. *For any modular lattice M and any elements $x, y, z \in M$, the following conditions are equivalent to each other and to their duals:*

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$

(ii) $x\pi \wedge (y\pi \vee z\pi) = (x\pi \wedge y\pi) \vee (x\pi \wedge z\pi)$

for any permutation π of the set $\{x, y, z\}$.

(iii) $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$

The purpose of this paper is to generalize Theorem B to a local version of Theorem A.

Definition 1. An ordered system $(y_0, y_1, \dots, y_{n+1})$ of elements of a modular lattice is called to be an n -distributive system if the following relation holds:

$$y_{n+1} \wedge \bigvee_{i=0}^n y_i = \bigvee_{j=0}^n [y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i].$$

This situation will be denoted by $D_n(y_0, y_1, \dots, y_{n+1})$. The system $(y_0, y_1, \dots, y_{n+1})$ is called n -tributive if the relation

$$M_n(y_0, y_1, \dots, y_{n+1}): \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i$$

holds. The relations dual to D_n (resp. M_n) will be denoted by D_n^* (resp. M_n^*).

Now we can formulate the two main results of this paper.

Theorem 1. *For any modular lattice M and for arbitrary elements*

$$y_0, y_1, \dots, y_{n+1} \in M$$

the following conditions are equivalent:

(A) $D_n(y_0, y_1, \dots, y_{n+1})$.

(B) $D_n(y_0\pi, y_1\pi, \dots, y_{n+1}\pi)$

for any permutation π of the set $\{y_0, y_1, \dots, y_{n+1}\}$.

(C) $M_n(y_0, y_1, \dots, y_{n+1})$.

Theorem 2. *Let M be a modular lattice and let $y_0, y_1, \dots, y_{n+1} \in M$. Set*

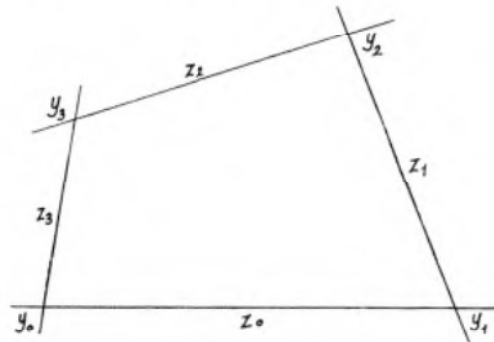
$$z_j = \bigvee_{\substack{i=0 \\ i \neq j-1, j-2 \pmod{n+2}}}^{n+1} y_i \quad (j = 0, 1, \dots, n+1).$$

Then the following conditions are equivalent:

(A) $D_n(y_0, y_1, \dots, y_{n+1})$.

(D) $D_n^*(z_0, z_1, \dots, z_{n+1})$.

Remark. The geometrical background of this theorem is the following obvious statement: If the four vertices of a square in a projective plane constitute a system of general position then the four sides of the square are also of general position and conversely (see Figure). Indeed, if the lattice M of Theorem 2 is the subspace lattice of a projective plane, then the above statement and Theorem 2 coincide.



To prove this theorems we need to generalize the technique used in [2]. First of all we generalize the notion of n -diamond (cf. [3]).

Definition 2. An ordered system $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1})$ of elements of a modular lattice is called a $\wedge$ - n -diamond if for any choice of the $(n+1)$ -element subsets $\{j_0, j_1, \dots, j_n\}$ and $\{k_0, k_1, \dots, k_n\}$ of the index set $\{0, 1, \dots, n+1\}$ the relation

$$\bar{b}_{j_0} \wedge \bigvee_{i=1}^n \bar{b}_{j_i} = \bar{b}_{k_0} \wedge \bigvee_{i=1}^n \bar{b}_{k_i}$$

holds. $(\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1})$ is called a $\vee$ - n -diamond if for any choice of the $(n+1)$ -element subsets $\{j_0, j_1, \dots, j_n\}$ and $\{k_0, k_1, \dots, k_n\}$ of the index set $\{0, 1, \dots, n+1\}$

$$\bigvee_{i=0}^n \underline{b}_{j_i} = \bigvee_{i=0}^n \underline{b}_{k_i}$$

holds. If $([\underline{b}_0, \bar{b}_0], [\underline{b}_1, \bar{b}_1], \dots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ is an ordered system of intervals of a modular lattice so that $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1})$ is a $\wedge$ - n -diamond and $(\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1})$ is a $\vee$ - n -diamond, then we call this system a generalized n -diamond. A generalized n -diamond (resp., $\wedge$ - n -diamond, $\vee$ - n -diamond) is called non-trivial if its elements are pairwise disjoint (resp., distinct).

The basic tool in proving Theorems 1 and 2 will be the following

Theorem 3. For any modular lattice M and elements $y_0, y_1, \dots, y_{n+1} \in M$ the following two conditions are equivalent:

- (I) $D_n(y_0, y_1, \dots, y_{n+1})$ does not hold.
- (II) There exists a non-trivial generalized n -diamond $([\underline{b}_0, \bar{b}_0], \dots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ in M such that $y_i \in [\underline{b}_i, \bar{b}_i]$ for $i=0, 1, \dots, n+1$.

2. Proof of Theorem 3

First we recall the following statement from [2].

Lemma 1. Let M be a modular lattice and let $y_0, y_1, \dots, y_{n+1} \in M$ such that

$$y_{n+1} \wedge \bigvee_{i=0}^n y_i \neq \bigvee_{j=0}^n [y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i].$$

Set

$$w_0 = y_{n+1} \wedge \bigvee_{i=0}^n y_i, \quad u_0 = \bigvee_{j=0}^n [y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i]$$

$$y'_j = \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i \quad (j = 0, 1, \dots, n)$$

$$v = \bigwedge_{j=0}^n (w_0 \vee y'_j), \quad b'_j = (y'_j \vee u_0) \wedge v \quad (j = 0, 1, \dots, n),$$

$$b_i = \bigwedge_{\substack{j=0 \\ j \neq i}}^n b'_j \quad (i = 0, 1, \dots, n), \quad u = \bigwedge_{j=0}^n b'_j, \quad w = u \vee w_0.$$

Then the elements b_i generate a Boolean sublattice B of length $n+1$ in B such that u is the g.l.b., v is the l.u.b. of B and w is a common relative complement of the dual atoms of B in $[u, v]$.

Lemma 2. The elements u and v defined in Lemma 1 can be written in the form

$$u = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i, \quad v = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i.$$

PROOF. Compute:

$$\begin{aligned} v &= \bigwedge_{j=0}^n (w_0 \vee y'_j) = \bigwedge_{j=0}^n [(y_{n+1} \wedge \bigvee_{i=0}^n y_i) \vee \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i] = \\ &= \bigwedge_{j=0}^n [(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i) \wedge \bigvee_{i=0}^n y_i] = \bigwedge_{j=0}^n \bigvee_{\substack{l=0 \\ l \neq j}}^{n+1} y_l \wedge \bigvee_{\substack{l=0 \\ l \neq n+1}}^{n+1} y_l = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i. \\ u &= \bigwedge_{j=0}^n b'_j = \bigwedge_{j=0}^n [(y'_j \vee u_0) \wedge v] = v \wedge \bigwedge_{j=0}^n (y'_j \vee u_0). \end{aligned}$$

Now we write $y'_j \vee u_0$ in a more suitable form.

$$\begin{aligned} y'_j \vee u_0 &= u_0 \vee y'_j = \bigvee_{k=0}^n [y_{n+1} \wedge \bigvee_{\substack{l=0 \\ l \neq k}}^n y_l] \vee \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i = \\ &= \bigvee_{\substack{k=0 \\ k \neq j}}^n [y_{n+1} \wedge \bigvee_{\substack{l=0 \\ l \neq k}}^n y_l] \vee \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i = \bigvee_{\substack{k=0 \\ k \neq j}}^n [(y_{n+1} \wedge \bigvee_{\substack{l=0 \\ l \neq k}}^n y_l) \vee \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i] = \\ &= \bigvee_{\substack{k=0 \\ k \neq j}}^n [(y_{n+1} \wedge \bigvee_{\substack{l=0 \\ l \neq k}}^n y_l) \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i \vee y_k] = \bigvee_{\substack{k=0 \\ k \neq j}}^n \{[(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge \bigvee_{\substack{l=0 \\ l \neq k}}^n y_l] \vee y_k\} = \\ &= \bigvee_{\substack{k=0 \\ k \neq j}}^n \{[(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge (y_j \vee \bigvee_{\substack{l=0 \\ l \neq j, k}}^n y_l)] \vee y_k\} = \\ &= \bigvee_{\substack{k=0 \\ k \neq j}}^n \{[(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge y_j] \vee [\bigvee_{\substack{l=0 \\ l \neq j, k}}^n y_l \vee y_k]\} = \\ &= \bigvee_{\substack{l=0 \\ l \neq j}}^n y_l \vee \bigvee_{\substack{k=0 \\ k \neq j}}^n [(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge y_j]. \end{aligned}$$

Then

$$u = v \wedge \bigwedge_{j=0}^n \left\{ \bigvee_{\substack{l=0 \\ l \neq j}}^n y_l \vee \bigvee_{\substack{k=0 \\ k \neq j}}^n [(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge y_j] \right\}.$$

In the above expression $\bigvee_{k=0, k \neq j}^n [(y_{n+1} \vee \bigvee_{i=0, i \neq j, k}^n y_i) \wedge y_j] \leq \bigvee_{l=0, l \neq j}^n y_l$ if $j \neq j'$ thus we can use the following form of the modular identity:

$$(1) \quad \bigvee_{i=1}^r p_i \vee \bigwedge_{i=1}^r q_i = \bigwedge_{i=1}^r (p_i \vee q_i)$$

provided $p_i \leq q_j$ if $i \neq j$. Hence we obtain

$$u = v \wedge \left\{ \bigwedge_{j=0}^n \bigvee_{\substack{l=0 \\ l \neq j}}^n y_l \vee \bigvee_{j=0}^n \bigwedge_{\substack{k=0 \\ k \neq j}}^n [(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \wedge y_j] \right\}.$$

Now write the join $\bigvee_{j=0}^n \bigvee_{k=0, k \neq j}^n [\dots]$ in the form $\bigvee_{k=0}^n \bigvee_{j=0, j \neq k}^n [\dots]$ and apply the dual of (1) to the join $\bigvee_{j=0, j \neq k}^n [\dots]$. Then

$$\begin{aligned} u &= v \wedge \left\{ \bigwedge_{j=0}^n \bigvee_{\substack{l=0 \\ l \neq j}}^n y_l \vee \bigvee_{k=0}^n \left[\bigvee_{j=0, j \neq k}^n y_j \wedge \bigwedge_{\substack{j=0 \\ j \neq k}}^n (y_{n+1} \vee \bigvee_{\substack{i=0 \\ i \neq j, k}}^n y_i) \right] \right\} = \\ &= v \wedge \left\{ \bigwedge_{j=0}^n \bigvee_{\substack{l=0 \\ l \neq j}}^n y_l \vee \bigvee_{k=0}^n \bigwedge_{\substack{j=0 \\ j \neq k}}^n \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i \right\} = v \wedge \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i = \\ &= \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i \wedge \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i, \end{aligned}$$

as claimed.

Lemma 3. Let M be a modular lattice, $y_0, y_1, \dots, y_{n+1} \in M$, and let $\bar{b}_i = y_i \vee u$, $\underline{b}_i = y_i \wedge v$ ($i=0, 1, \dots, n+1$). Then $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1})$ is a $\wedge$ - n -diamond and g.l.b. $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\} = u$, $(\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1})$ is a $\vee$ - n -diamond and l.u.b. $\{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\} = v$.

PROOF. It suffices to show that

$$(2) \quad (y_0 \vee u) \wedge \left(\bigvee_{i=1}^n (y_i \vee u) \right) = u$$

and

$$(3) \quad \bigvee_{i=0}^n (y_i \wedge v) = v.$$

PROOF OF (2). First we compute $\bigwedge_{j, j \neq k} \bigvee_{i, i \neq j, k} y_i$ for all $k (=0, 1, \dots, n+1)$.

(a) If $k=0$, then by the dual of (1)

$$\bigwedge_{\substack{j=0 \\ j \neq 0}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, 0}}^{n+1} y_i = \bigvee_{i=0}^{n+1} y_i \wedge \bigwedge_{\substack{j=0 \\ j \neq 0, n+1}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, 0}}^{n+1} y_i = \bigvee_{j=0}^{n+1} [y_j \wedge \bigvee_{\substack{i=0 \\ i \neq 0, j}}^{n+1} y_i].$$

(b) If $k=n+1$, then similarly

$$\bigwedge_{\substack{j=0 \\ j \neq n+1}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, n+1}}^{n+1} y_i = \bigvee_{\substack{j=0 \\ j \neq 0, n+1}}^{n+1} [y_j \wedge \bigvee_{\substack{i=0 \\ i \neq j, n+1}}^{n+1} y_i].$$

(c) If $k \neq 0, n+1$, then

$$\begin{aligned} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i &= \bigvee_{\substack{j=0 \\ j \neq k, n+1}}^{n+1} [y_j \wedge \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i] = \\ &= \bigvee_{\substack{j=0 \\ j \neq 0, k, n+1}}^{n+1} [y_j \wedge \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i] \vee [y_0 \wedge \bigvee_{\substack{i=0 \\ i \neq 0, k}}^{n+1} y_i]. \end{aligned}$$

(d) Let $y_j^{(k)} = y_j \wedge \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i$. Then, by Lemma 2,

$$\begin{aligned} u &= \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i = \\ &= \left[\bigvee_{\substack{j=0 \\ j \neq 0 \\ j \neq n+1}}^{n+1} y_j^{(0)} \vee \bigvee_{\substack{j=0 \\ j \neq 0 \\ j \neq n+1}}^{n+1} y_j^{(n+1)} \vee \bigvee_{\substack{k=0 \\ k \neq 0 \\ k \neq n+1}}^{n+1} \bigvee_{\substack{j=0 \\ j \neq 0 \\ j \neq k}}^{n+1} y_j^{(k)} \right] \vee \bigvee_{\substack{k=0 \\ k \neq 0 \\ k \neq n+1}}^{n+1} y_0^{(k)}. \end{aligned}$$

Now let this latter join be abbreviated by $A \vee B$. Then it is easy to see that $A \equiv \bigvee_{j, j \neq 0, j \neq n+1}^{n+1} y_j$ and $B \equiv y_0$. Thus again by the dual of (1)

$$\begin{aligned} (y_0 \vee A \vee B) \wedge \left(\bigvee_{\substack{j=0 \\ j \neq 0, n+1}}^{n+1} y_j \vee A \vee B \right) &= (y_0 \vee A) \wedge \left(\bigvee_{\substack{j=0 \\ j \neq 0, n+1}}^{n+1} y_j \vee B \right) = \\ &= (y_0 \wedge \bigvee_{\substack{j=0 \\ j \neq 0, n+1}}^{n+1} y_j) \vee A \vee B = y_0^{(n+1)} \vee A \vee B = y_0^{(n+1)} \vee u = u, \end{aligned}$$

since for all $y_j^{(k)}, y_j^{(k)} \leq u$.

PROOF OF (3).

$$\begin{aligned} \bigvee_{\substack{i=0 \\ i \neq n+1}}^{n+1} (y_i \wedge v) &= \bigvee_{\substack{i=0 \\ i \neq n+1}}^{n+1} (y_i \wedge \bigwedge_{\substack{j=0 \\ j \neq i}}^{n+1} \bigvee_{\substack{l=0 \\ l \neq j}}^{n+1} y_l) = \bigvee_{\substack{i=0 \\ i \neq n+1}}^{n+1} (y_i \wedge \bigvee_{\substack{l=0 \\ l \neq i}}^{n+1} y_l) = \\ &= \bigvee_{\substack{i=0 \\ i \neq n+1}}^{n+1} y_i \wedge \bigwedge_{\substack{i=0 \\ i \neq n+1}}^{n+1} \bigvee_{\substack{l=0 \\ l \neq i}}^{n+1} y_l = \bigwedge_{\substack{i=0 \\ i \neq n+1}}^{n+1} \bigvee_{\substack{l=0 \\ l \neq i}}^{n+1} y_l = v. \end{aligned}$$

This completes the proof of the Lemma.

PROOF OF THEOREM 3, (I) $\Rightarrow$ (II). The intervals $[b_0, \bar{b}_0], [b_1, \bar{b}_1], \dots, [b_{n+1}, \bar{b}_{n+1}]$, formed from the elements $b_0, b_1, \dots, b_{n+1}$ and $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}$ defined in Lemma 3, satisfy the conditions given in (II), provided they are disjoint. Assume that there exist integers $l, m \in \{0, 1, \dots, n+1\}$ such that $[b_l, \bar{b}_l] \cap [b_m, \bar{b}_m] \neq \emptyset$. Then $b_l \leq \bar{b}_l \wedge \bar{b}_m = u$. Hence, computing by Lemma 3,

$$\begin{aligned} v &= \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} b_i = \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \left(\bigvee_{\substack{i=0 \\ i \neq j, l}}^{n+1} b_i \vee b_l \right) \wedge \bigvee_{\substack{i=0 \\ i \neq l}}^{n+1} b_i \leq \\ &\leq \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \left(\bigvee_{\substack{i=0 \\ i \neq j, l}}^{n+1} b_i \vee b_l \right) \leq \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \left(\bigvee_{\substack{i=0 \\ i \neq j, l}}^{n+1} \bar{b}_i \vee u \right) = \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, l}}^{n+1} \bar{b}_i = u. \end{aligned}$$

This yields $v = u$ which is a contradiction since, by Lemma 1, the length of the interval $[u, v]$ is at least $n+1$. Q.e.d.

PROOF OF THEOREM 3, (II) $\Rightarrow$ (I). Let $\bar{u} = \text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}$ and let $v = \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}$. Assume that both (II) and $\neg$ (I) hold. Then

$$\begin{aligned} b_{n+1} &= b_{n+1} \wedge v = b_{n+1} \wedge \bigvee_{i=0}^n b_i \leq y_{n+1} \wedge \bigvee_{i=0}^n y_i = \\ &= \bigvee_{j=0}^n [y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n y_i] \leq \bigvee_{j=0}^n [\bar{b}_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^n \bar{b}_i] = \bar{u}. \end{aligned}$$

Hence, by a similar computation as in the proof of (I) $\Rightarrow$ (II) we obtain $v \leq \bar{u}$. This yields that $b_i \leq v \leq \bar{u} \leq \bar{b}_i$ which contradicts the assumption that the intervals $[b_i, \bar{b}_i]$ are disjoint. Q.e.d.

3. Proof of Theorems 1 and 2

PROOF OF THEOREM 1. (A) $\Leftrightarrow$ (B). is obvious from Theorem 3.

(B) $\Rightarrow$ (C) has been proved in [2] (see the proof of [2], Corollary 2.3).

$\neg$ (A) $\Rightarrow$ $\neg$ (C). Assume that $D_n(y_0, y_1, \dots, y_{n+1})$ does not hold. Then, by Theorem 3, there is a non-trivial generalized n -diamond $([b_0, \bar{b}_0], [b_1, \bar{b}_1], \dots, [b_{n+1}, \bar{b}_{n+1}])$ in M such that $b_i \leq y_i \leq \bar{b}_i$ ($i=0, 1, \dots, n+1$). Now assume that $M_n(y_0, y_1, \dots, y_{n+1})$ holds. Then we obtain that

$$\begin{aligned} \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\} &= \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} b_i \leq \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i = \\ &= \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i \leq \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} \bar{b}_i = \text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}, \end{aligned}$$

a similar contradiction as in the proof of Theorem 3, (II) $\Rightarrow$ (I).

PROOF OF THEOREM 2. Define y_i and z_j as in the Introduction and set

$$\bar{y}_i = \bigwedge_{\substack{j=0 \\ j \neq i+1, i+2 \pmod{n+2}}}^n z_j \quad (i = 0, 1, \dots, n+1).$$

First we prove the following statement:

(a) $D_n(y_0, y_1, \dots, y_{n+1})$ implies $D_n(\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+1})$.

Indeed, $\bar{y}_i \leq z_j$ if $j \neq i+1, i+2 \pmod{n+2}$, thus $\bar{y}_i \leq z_j$ if $i \neq j-1, j-2 \pmod{n+2}$. Hence

$$\bigvee_{\substack{i=0 \\ i \neq j-1, j-2 \pmod{n+2}}}^{n+1} \bar{y}_i \leq z_j = \bigvee_{\substack{i=0 \\ i \neq j-1, j-2 \pmod{n+2}}}^{n+1} y_i \quad (j = 0, 1, \dots, n+1),$$

thus

$$\bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} \bar{y}_i \leq \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i$$

holds for any $j (= 0, 1, \dots, n+1)$. Also $y_i \leq \bar{y}_i$, whence

$$\bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} y_i \leq \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j, k}}^{n+1} \bar{y}_i \leq \bigvee_{j=0}^{n+1} \bigwedge_{\substack{i=0 \\ i \neq j}}^{n+1} \bar{y}_i \leq \bigvee_{j=0}^{n+1} \bigwedge_{\substack{i=0 \\ i \neq j}}^{n+1} y_i.$$

This proves (a).

Therefore, it suffices to show the following two statements:

(b) $\Gamma D_n(y_0, y_1, \dots, y_{n+1})$ implies $\Gamma D_n^*(z_0, z_1, \dots, z_{n+1})$.

(c) $\Gamma D_n^*(z_0, z_1, \dots, z_{n+1})$ implies $\Gamma D_n(\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n+1})$.

But, by symmetry and duality, (c) follows from (b), thus it is sufficient to prove only (b).

Now assume that $D_n(y_0, y_1, \dots, y_{n+1})$ does not hold. Then there is a non-trivial generalized n -diamond $([b_0, \bar{b}_0], [b_1, \bar{b}_1], \dots, [b_{n+1}, \bar{b}_{n+1}])$ in M such that $b_i \leq y_i \leq \bar{b}_i$. We finish the proof in the following three steps.

1. Obviously $z_j \leq \bigvee_{i=0, i \neq j-1, j-2 \pmod{n+2}}^{n+1} \bar{b}_i$. Hence, computing in the generalized n -diamond, we obtain that $\bigwedge_{j=0, j \neq k}^{n+1} z_j \leq \text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}$. Thus

$$\bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} z_j \leq \text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}.$$

2. We prove that $\bigwedge_{l=0}^{n+1} \bigvee_{k=0, k \neq l}^{n+1} \bigwedge_{j=0, j \neq k, l}^{n+1} z_j \leq \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}$. It is easy to show (see, for instance, the proof of [2] Corollary 2.3), that

$$\bigwedge_{k=0}^{n+1} \bigvee_{\substack{j=0 \\ j \neq k}}^{n+1} \bigwedge_{\substack{i=0 \\ i \neq j, k}}^{n+1} z_i = \bigwedge_{j=0}^{n+1} \bigwedge_{\substack{k=0 \\ k \neq j}}^{n+1} [z_j \vee \bigwedge_{\substack{i=0 \\ i \neq j, k}}^{n+1} z_i].$$

Thus, by symmetry, it suffices to show that

$$z_0 \vee \bigwedge_{\substack{j=1 \\ j \neq l}}^{n+1} z_j \cong \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}$$

holds for all $l(=1, 2, \dots, n+1)$. This is trivial if $l=1$ or $l=n+1$. Indeed, if, for instance, $l=1$, then

$$z_0 \vee \bigwedge_{j=2}^{n+1} z_j \cong \bigvee_{i=0}^{n-1} y_i \vee y_{n+1} \cong \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}.$$

Assume that $l \neq 1, n+1$. Then

$$\begin{aligned} z_0 \vee \bigwedge_{\substack{j=1 \\ j \neq l}}^{n+1} z_j &= z_0 \vee \left[\bigwedge_{j=1}^{l-1} z_j \wedge \bigwedge_{j=l+1}^{n+1} z_j \right] \cong \\ &\cong \bigvee_{i=0}^{n-1} y_i \vee \left[\bigvee_{i=l-1}^n y_i \wedge \left(\bigvee_{i=0}^{l-2} y_i \vee y_{n+1} \right) \right] = \\ &= \bigvee_{i=0}^{l-2} y_i \vee \bigvee_{i=l-1}^{n-1} y_i \vee \left[\bigvee_{i=l-1}^n y_i \wedge \left(\bigvee_{i=0}^{l-2} y_i \vee y_{n+1} \right) \right] = \\ &= \left[\bigvee_{i=0}^{l-2} y_i \vee \bigvee_{i=l-1}^n y_i \right] \wedge \left[\bigvee_{i=l-1}^{n-1} y_i \vee \bigvee_{i=0}^{l-2} y_i \vee y_{n+1} \right] \cong \\ &\cong \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}, \end{aligned}$$

as claimed.

3. Because of the two previous observations $D_n^*(z_0, z_1, \dots, z_{n+1})$ would mean that $\text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\} = \text{l.u.b. } \{b_0, b_1, \dots, b_{n+1}\}$, i.e., $\text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\} \in [b_i, \bar{b}_i]$ for all i ($i=0, 1, \dots, n+1$). This contradicts the fact that the intervals $[b_i, \bar{b}_i]$ are disjoint. Thus $D_n^*(z_0, z_1, \dots, z_{n+1})$ does not hold which completes the proof.

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