Holomorphy theory of F loops

By F. LEONG (U.S.M., Malaysia)

D. A. ROBINSON has written "Holomorphy Theory of Extra Loops" [6]. The present paper extends the results in his paper to a class of loops much wider than that of extra loops. All terms and symbols follow those in [2], [3] unless otherwise stated.

Let G be a Moufang loop. G_a is defined as the subloop generated by all the associators (x, y, z) of G and G_c is defined as the subloop generated by all the commutators $[x, y]$ of G. G_a is normal in G, [5].

An F loop is a Moufang loop such that if H is a subloop generated by any three elements, then $H_a \subset Z(H)$ where $Z(H)$ is the centre of H.

An EF loop G is an F loop such that G_a is nilpotent.

A pE loop G is a Moufang loop such that G/N is commutative of exponent p, p a prime and N is the nucleus of G.

It is known that extra loops [4] and commutative Moufang loops [1] are $2E$ loops and $3E$ loops respectively. We also have the following:

Result: *If G is a Moufang loop such that G/N is commutative, then G is an EF loop.*

PROOF. As G/N is commutative, $G_c \subset N$. By [2], p. 125, L 5.5, G is an F loop. By [2], p. 161, T. 11.4, $(G/N)' = G'N/N$ is of exponent 3. Thus, $(G'N)^3 = N$ and $G_a^3 \subset N$. By [3], $G_a^3 \subset Z(G_a)$. As G_a/G_a^3 is a 3-loop, it is nilpotent. Thus G_a is nilpotent and G is an EF loop.

Definition. Let G be a loop and $\mathscr{A}$ a group of automorphisms of G. Let $H = (\mathscr{A}, G) = \{(A, a) \mid A \in \mathscr{A}, a \in G\}$. Multiplication on H is defined as: $(A, a)(B, b) = (AB, aBb)$. Then, H is a loop which is called the $\mathscr{A}$ holomorph of G. See [1], p. 336, Sec. 5.

Lemma 1. *Let G be a loop and $\mathscr{A}$ a group of automorphisms. Let $H = (\mathscr{A}, G)$ be the $\mathscr{A}$ holomorph of G. Writing*

$$(A, a)(B, b) = \big((B, b)(A, a)\big)(X, x) \ldots\ldots\ldots\ldots\ldots (\mathrm{R})$$

$$(A, a)(B, b)\cdot(C, c) = \big((A, a)\cdot(B, b)(C, c)\big)(Y, y) \ldots (\mathrm{S})$$

Then,

$$X = [A, B],\ aBb = (bAa)[A, B]\cdot x$$

$$Y = I,\ y = (aBC, bC, c).$$

PROOF. By definition and computation.

Definition. Let G be an I. P. loop. Let $M[G]$, the Moufang nucleus of G, be defined as the set $\{g|g\in G, gx\cdot yg=(g\cdot xy)g, \forall x, y\in G\}$. It is proved in [1], p. 297, $T.4A$ that $M[G]$ is a Moufang loop.

Proposition 1. *Let $H=(\mathcal{A}, G)$ where $\mathcal{A}$ is a group of automorphisms of the Moufang loop G. Then*

(a) $M[H]=\{(A, a)|A\in\mathcal{A}, a\in G, a^{-1}\cdot aB\in N(G)\forall B\in\mathcal{A}\}$.

(b) *G is an F loop$\Rightarrow M[H]$ is an F loop.*

(c) *G is an EF loop$\Rightarrow M[H]$ is an EF loop.*

(d) *G is a pE loop$\Rightarrow M[H]$ is a pE loop.*

PROOF.

(a) By [1], p. 336, T.5A, (v).

(b) Let $(A, a), (B, b), (C, c)\in M[H]$. By (a) and lemma 1 (R), $[(B, b), (C, c)]=([B, C], [b, c]n_0)$, $n_0\in N$. By (a) and lemma 1 (S), $((A, a), (B, b), [(B, b), (C, c)])=(I, (an_1, bn_2, [b, c]n_0))$, $n_1, n_2\in N$, $=(I, (a, b, [b, c]))=(I, 1)$ as G is an F loop.

(c) By (a) and lemma 1 (S), every associator of $M[H]$ is of the form $(1, (a, b, c))$, $a, b, c\in G$. Conversely, every element $(1, (a, b, c))$ is in H_a since it is equal to $((1, a), (1, b), (1, c))$. Thus $M[H]_a\cong G_a$. Since G_a is nilpotent, $(M[H])_a$ is nilpotent.

(d) Let $(A, a), (B, b), (C, c), (D, d)\in M[H]$. By (a) and lemma 1 (R), $[(A, a), (B, b)]=([A, B], [a, b]n)$, $n\in N=([A, B], n_0)$, $n_0\in N$, as G is an pE loop. By (a) and lemma 1 (S), $((C, c), (D, d), [(A, a), (B, b)])=(I, (cn_1, dn_2, n_0))$ ease; $=(I, 1)$ $n_1, n_2\in N$.

Also, by (a), $(A, a^p)=(A^p, a^p n_0)$, $n_0\in N$. As $a^p\in N$, $(A, a)^p\in N(M[H])$. Thus, $M[H]$ is a pE loop.

Proposition 2. *Let $\mathcal{A}$ be a group of automorphisms of a loop G. Let $H=(\mathcal{A}, G)$. Then $gA\in gN\forall A\in\mathcal{A}$, $\forall g\in G$ together with G being*

(a) *an F loop$\Leftrightarrow H$ is an F loop;*

(b) *an EF loop$\Leftrightarrow H$ is an EF loop;*

(c) *a pE loop$\Leftrightarrow H$ is an pE loop.*

PROOF. (a) Suppose H is an F loop. Then, G is an F loop since it is isomorphic to a subloop (I, G) of H. Since H is Moufang, $M[H]=H$. By Proposition 1 (a), $gA\in gN\forall g\in G$, $\forall A\in\mathcal{A}$.

Conversely, suppose G is an F loop with $gA\in gN\forall g\in G$, $\forall A\in\mathcal{A}$. Then, $M[H]=H$ by Proposition 1 (a) and H is an F loop by Proposition 1 (b). The other cases are similarly treated.

Remark. (c) is a generalization of the Main Theorem in [6].

Lemma 2. *Let G be a Moufang loop such that $G_c\subset N$ and let* Aut (G) *be the group of automorphisms of G. Let*

$$\mathcal{A} = \{A|A\in \text{Aut}\,(G), gA\in gN, \forall g\in G\}.$$

Then, either $\mathcal{A}$ is a nontrivial subgroup of Aut (G) *or* $|G|\leqq 2$.

PROOF. By [5], as $G_c \in N$, $R^3(x, y) \in \mathrm{Aut}\,(G)$, $x, y \in G$. As shown before, $G_a^2 \subset N$. Then, $zR^3(x, y) = z(z, x, y)^3 \in zG_a^3 \subset zN$, $\forall z \in G$. So, $R^3(x, y) \in \mathscr{A}$.

Suppose $R^3(x, y) = 1\, \forall x, y \in G$. Then $(z, x, y)^3 = 1\, \forall z \in G$. Thus., $z^3 \in N$. Hence G/N is a commutative Moufang loop of exponent 3. By [1], p. 302, (iv), $I(G) \subset \subset \mathrm{Aut}\,(G)$. In particular, $T(x) \in \mathrm{Aut}\,(G)\, \forall x \in G$. As $G_c \subset N$, $T(x) \in \mathscr{A}$ since $gT(x) = = g[g, z]\, \forall g \in G$.

Suppose $T(x) = I\, \forall x \in G$. Then G is a commutative Moufang loop. Let $z_2 \in Z_2 - Z$. Then, $R(z_2, w) \neq I$ for some $w \in G$. Hence, $xR(z_2, w) = x(x, z_2, w) \in xZ = = xN\, \forall x \in G$. This implies that $R(z_2, w) \in \mathscr{A}$. If $Z_2 = Z$, then G is an abelian group. Then, $\mathscr{A} = \mathrm{Aut}\,(G) \neq 1$ unless $G = C_2$ or 1.

Corollary: *$\exists$ a strictly increasing sequence of EF loops.*

Let G be any nonassociative Moufang loop such that $G_c \subset N$. By our result, it is an EF loop. Let $H = (\mathscr{A}, G)$ where $\mathscr{A}$ is defined by lemma 2. It is easy to see that $H_c \subset N(H)$. Continue with H.

Corollary: *$\exists$ a nonassociative pE loop of order p^n for any $n \geqq 4$ if $p \leqq 3$ and any $n \geqq 5$ if $p > 3$.*

PROOF. For $p \leqq 3$; the existence of a non-associative Moufang loop G of order p^4 is well known. It is clear that $\exists x, y \in G$ such that $R(x, y)$ is an automorphism of G of order p and that $gR(x, y) \in gN\, \forall g \in G$. Let $\mathscr{A} = \langle R(x, y) \rangle$; $H = (\mathscr{A}, G)$. By Proposition 2 (c) H is a pE loop of order p^5. As $H/(I, G) \cong (\mathscr{A}, 1)$, $H_a \cong G_a \cong C_p$. Continue with H.

For $p \geqq 5$: By [7], p. 408, those Moufang loops constructed can be verified to be nonassociative pE loops of order p^5. Proceed as above.

References

[1] R. H. BRUCK, Contribution to the Theory of Loops, *Trans. A.M.S.* **60** (1946), 245—354.

[2] R. H. BRUCK, A survey of Binary System, *Berlin—New York*, 1971.

[3] F. FENYVES, Extra Loops I, *Publ. Math. (Debrecen)* **15** (1968), 235—238.

[4] F. FENYVES, Extra Loops II. On Loops with Identities of Bol-Moufang Type, *Publ. Math. (Debrecen)* **16** (1969), 187—192.

[5] F. LEONG, The Devil and Angel of Loops, *Proceedings A.M.S.* **54** (1976), 32—34.

[6] D. A. ROBINSON, Holomorphy Theory of Extra Loops, *Publ. Math. (Debrecen)* **18** (1971), 59—64.

[7] C. R. B. WRINGT, Nilpotency Conditions for Finite Loops, *Illinois J. Math.* **9** (1965), 399—409.

(Received June 16, 1977.)