

Decomposition of curvature tensor of second kind in a birecurrent areal space

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1. *Introduction.* We consider the areal space $A_n^{(m)}$ (n : the dimension of the space; m : the order of the integral) in which the areal metric tensor is defined as ([2], p. 289)¹⁾

$$(1.1) \quad g_{ij}^{\alpha\beta}(x^h, \dot{x}_\epsilon^h) = \left[\frac{m}{2} \frac{\partial^2 \{L(x^h, \dot{x}_\epsilon^h)\}^{\frac{2}{m}}}{\partial \dot{x}_\alpha^i \partial \dot{x}_\beta^j} \right],$$

where²⁾ $L = L(x^h, \dot{x}_\epsilon^h)$ satisfies the relation

$$(1.2) \quad \frac{\partial L}{\partial \dot{x}_\alpha^i} \dot{x}_\alpha^i = mL.$$

We have

$$(1.3) \quad g_{ij}^{\alpha\beta} \dot{x}_\alpha^i \dot{x}_\beta^j = mL^{\frac{2}{m}} \quad \text{and} \quad g_{ik}^{\alpha\gamma} \dot{x}_\alpha^i = L^{\left(\frac{2}{m}-1\right)} \frac{\partial L}{\partial \dot{x}_\gamma^k}.$$

The covariant derivative of a vector field $X_\epsilon^i(x^h, \dot{x}_\epsilon^h)$ with respect to x^j is given by ([1])

$$(1.4) \quad X_{\epsilon|j}^i = \frac{\partial X_\epsilon^i}{\partial x^j} - \frac{\partial X_\epsilon^i}{\partial \dot{x}_\lambda^1} \Gamma_{pj}^i \dot{x}_\lambda^p + \Gamma_{j\lambda}^i X_\epsilon^1,$$

where $\Gamma_{j\lambda}^i$ is a connection coefficient.

The commutation formula for a vector field $X_\epsilon^i(x^h, \dot{x}_\epsilon^h)$ involving the covariant derivative of the type (1.4) is given by ([1])

$$(1.5) \quad X_{\epsilon|k|h}^i - X_{\epsilon|h|k}^i = X_\epsilon^j K_{2jkh}^i - \frac{\partial X_\epsilon^i}{\partial \dot{x}_\alpha^1} K_{1pkh}^1 \dot{x}_\alpha^p + X_{\epsilon|j}^i T_{kh}^j,$$

where K_{1jkh}^i and K_{2jkh}^i are curvature tensors of first and second kinds respectively and

$$(1.6) \quad T_{kh}^i = \Gamma_{kh}^i - \Gamma_{hk}^i.$$

¹⁾ The numbers in brackets refer to the references at the end of the paper.

²⁾ Throughout this paper the Latin indices $i, j, k \dots$ run over 1 to n , Greek indices $\alpha, \beta, \gamma \dots$ run over 1 to m and $\dot{x}_\epsilon^h = \frac{\partial x^h}{\partial t^\epsilon}$.

The Bianchi identities corresponding to the curvature tensors of first and second kinds are given by ([1])

$$(1.7) \quad K_{jkh|l}^i + \frac{\partial \Gamma_{kj}^i}{\partial \dot{x}_\alpha^p} K_{qhl}^p \dot{x}_\alpha^q + T_{kh}^q K_{jql}^i + \text{Cycl}(k, l, h) = 0$$

where $\text{cycl}(k, l, h)$ denotes the sum of two sets of the three terms each obtained by replacing the indices (k, l, h) by first (l, h, k) and then by (h, k, l) in the expression on the left hand side of (1.7). The curvature tensor of second kind satisfies the recurrent properties of first and second order as

$$(1.8) \quad K_{jkh|l}^i = v_l K_{jkh}^i$$

and

$$(1.9) \quad K_{jkh|l|m}^i = a_{lm} K_{jkh}^i,$$

where v_l and a_{lm} are recurrence vector and tensor fields. Under the decomposition

$$(1.10) \quad K_{jkh}^i = \dot{x}_\alpha^i \Phi_{jkh}^\alpha,$$

the decomposed tensor field Φ_{jkh}^α behaves like a recurrent tensor field:

$$(1.11) \quad \Phi_{jkh|l}^\alpha = W_l \Phi_{jkh}^\alpha,$$

where

$$(1.12) \quad W_l = v_l - m^{-1} L^{-\frac{2}{m}} g_{li}^\alpha \dot{x}_\alpha^i \dot{x}_{\alpha|l}^i.$$

2. Decomposition of K_{jkh}^i in birecurrent areal space.

Let us consider the decomposition of the curvature tensor of second kind as

$$(2.1) \quad K_{jkh}^i = \dot{x}_\alpha^i \psi_{jkh}^\alpha,$$

where $\psi_{jkh}^\alpha = \psi_{jkh}^\alpha(x^i, \dot{x}_\alpha^i)$ is a non-zero homogeneous tensor field of degree—1 in its directional arguments and is called as the decomposition tensor field.

Theorem 2.1. *The decomposition tensor field ψ_{jkh}^α behaves like recurrent curvature tensor field of second order:*

$$(2.2) \quad \psi_{jkh|l|m}^\alpha = A_{lm} \psi_{jkh}^\alpha,$$

where

$$(2.3) \quad A_{lm} = a_{lm} - (\dot{x}_{\alpha|l|m}^i + W_m \dot{x}_{\alpha|l}^i + W_l \dot{x}_{\alpha|m}^i) m^{-1} L^{-1} \frac{\partial L}{\partial \dot{x}_\alpha^i}.$$

PROOF. Differentiating (2.1) covariant with respect to x^l, x^m , successively and taking into account the relations (1.9) and (2.1), we get

$$(2.4) \quad a_{lm} \dot{x}_\alpha^i \psi_{jkh}^\alpha = \dot{x}_{\alpha|l|m}^i \psi_{jkh}^\alpha + \dot{x}_{\alpha|l}^i \psi_{jkh|m}^\alpha + \dot{x}_{\alpha|m}^i \psi_{jkh|l}^\alpha + \dot{x}_\alpha^i \psi_{jkh|l|m}^\alpha.$$

Multiplying the relation (2.4) by $g_{li}^{\delta\alpha} \dot{x}_\delta^i$, we have

$$(2.5) \quad a_{lm} g_{li}^{\delta\alpha} \dot{x}_\delta^i \dot{x}_\alpha^j \psi_{jkh}^\alpha = \dot{x}_{\alpha|i|l}^i g_{li}^{\delta\alpha} \dot{x}_\delta^j \psi_{jkh}^\alpha + \\ + \dot{x}_{\alpha|i}^i \psi_{jkh|l}^\alpha g_{li}^{\delta\alpha} \dot{x}_\delta^j + \dot{x}_{\alpha|l}^i \psi_{jkh|i}^\alpha g_{li}^{\delta\alpha} \dot{x}_\delta^j + \dot{x}_\alpha^i \psi_{jkh|i|l}^\alpha g_{li}^{\delta\alpha} \dot{x}_\delta^j.$$

In the light of the relations (1.11) and (1.3), the relation (2.5) yields the theorem (2.1).

Corollary 2.1. *The recurrent tensor fields of curvature and decomposed tensor fields are not equal.*

PROOF. Obvious from the relations (2.2) and (2.3).

Theorem 2.2. *Under the decomposition (2.1), the following identity holds:*

$$(2.6) \quad K_{jk[h|i|l|m]}^i = \dot{x}_\alpha^i (a_{lm} \psi_{jkh}^\alpha + a_{mh} \psi_{jkl}^\alpha + a_{hl} \psi_{jkm}^\alpha).$$

PROOF. Adding the expressions obtained by cyclic change of the indices h, l, m and taking into account the relation (2.1), we have the required identity.

Theorem 2.3. *Under the decomposition (2.1) the relation*

$$(2.7) \quad \{(A_{lm} - A_{ml})_{|n} - (A_{lm} - A_{ml})W_n + W_s W_n T_{lm}^s - W_{s|n} T_{lm}^s - W_s T_{lm|n}^s\} \psi_{jkh}^\alpha + \\ + \dot{x}_{\alpha|n}^s (\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) = 0$$

holds provided ψ_{jkh}^α is independent of direction.

PROOF. Commuting (2.2) with respect to the indices l , and m , we get

$$(2.8) \quad (A_{lm} - A_{ml}) \psi_{jkh}^\alpha = \psi_{jkh|l|m}^\alpha - \psi_{jkh|m|l}^\alpha.$$

With aid of the commutation formula (1.5), the relations (2.1) and (1.11), (2.8) becomes

$$(2.9) \quad (A_{lm} - A_{ml}) \psi_{jkh}^\alpha = -\dot{x}_\alpha^s (\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) - \\ - \frac{\partial \psi_{jkh}^\alpha}{\partial x_\alpha^p} K_{1q|lm}^p \dot{x}_\alpha^q + W_s \psi_{jkh}^\alpha T_{lm}^s.$$

Differentiating (2.9) covariantly with respect to x^n and by virtue of (1.11), we get

$$(2.10) \quad (A_{lm} - A_{ml})_{|n} \psi_{jkh}^\alpha + (A_{lm} - A_{ml})W_n \psi_{jkh}^\alpha = -\dot{x}_{\alpha|n}^s (\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) - \\ - 2W_n \dot{x}_\alpha^s (\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) - \left(\frac{\partial \psi_{jkh}^\alpha}{\partial x_\alpha^p} K_{1q|lm}^p \dot{x}_\alpha^q \right)_{|n} + (W_s \psi_{jkh}^\alpha T_{lm}^s)_{|n}.$$

In the light of the relation (2.9) and the condition imposed on ψ_{jkh}^α , a little simplification of the relation (2.10) yields the theorem (2.3).

Theorem 2.4. Under the decomposition (2.1), if W_l is independent of $\dot{x}_a^i$, the relation

$$(2.11) \quad \{(A_{[lm|n]} - A_{[ln|m]}) + (A_{[lm|w_n]} - A_{[ln|w_m]}) - A_{[l(s)T_{mn}^s]}\} \psi_{jkh}^a = \\ = -\dot{x}_a^s \{W_{[l} \psi_{j]mn}^s \psi_{skh}^a + W_{[l} \psi_{j]mn}^s \psi_{jsh}^a + W_{[l} \psi_{j]mn}^s \psi_{jks}^a + W_s \psi_{jks}^a \psi_{lmn}^a\} + \\ + \frac{\partial \psi_{jkh}^a}{\partial \dot{x}_a^p} W_{[l} K_{1(q)mn}^p \dot{x}_a^q$$

holds³⁾.

PROOF. In the light of (1.11) the covariant differentiation of (2.2) with respect to x^n yields

$$(2.12) \quad \psi_{jkh|l|m|n}^a = (A_{lm|n} + A_{lm} w_n) \psi_{jkh}^a.$$

Commuting the indices m, n in (2.12), we have

$$(2.13) \quad \psi_{jkh}^a \{(A_{lm|n} - A_{ln|m}) + (A_{lm} w_n - A_{ln} w_m)\} = \psi_{jkh|l|m|n}^a - \psi_{jkh|l|n|m}^a.$$

With the help of commutation formula (1.5) and the relations (1.11) and (2.1), (2.13) yields

$$(2.14) \quad \{(A_{lm|n} - A_{ln|m}) + (A_{lm} w_n - A_{ln} w_m)\} \psi_{jkh}^a = \\ = -W_l \psi_{skh}^a \dot{x}_a^s \psi_{jmn}^a - W_l \psi_{jsh}^a \dot{x}_a^s \psi_{kmn}^a - \\ - W_l \psi_{jks}^a \dot{x}_a^s \psi_{hmn}^a - W_s \psi_{jkh}^a \dot{x}_a^s \psi_{lmn}^a - \frac{\partial W_l}{\partial \dot{x}_a^p} \psi_{jkh}^a \\ K_{1qmn}^p \dot{x}_a^q - \frac{\partial \psi_{jkh}^a}{\partial \dot{x}_a^p} W_l K_{1qmn}^p \dot{x}_a^q + A_{ls} \psi_{jkh}^a T_{mn}^s.$$

Changing the indices l, m , and n cyclically in (2.14) and adding the obtained results, we get the theorem (2.4).

Theorem 2.5. Under the decomposition (2.1), if connection coefficient is independent of direction, the relation

$$(2.15) \quad \psi_{j[kh}^a a_{l]m} + \psi_{jq[l}^a T_{kh]m}^q + v_m \psi_{jq[l}^a T_{kh]}^q = 0$$

holds.

PROOF. Differentiating (1.7) covariantly with respect to x^m and taking into account the condition imposed on Γ_{jk}^i and the relations (1.8), (1.9), we have

$$(2.16) \quad a_{lm} K_{2jkh}^i + T_{kh|m}^q K_{2jq}^i + T_{kh}^q v_m K_{2jq}^i + a_{hm} K_{2jlk}^i + T_{lk|m}^q K_{2jqh}^i + \\ + T_{lk}^q v_m K_{2jqh}^i + a_{km} K_{2jhl}^i + T_{hl|m}^q K_{2jqk}^i + T_{hl}^q v_m K_{2jqk}^i = 0.$$

In the light of (2.1), a little simplification of (2.16) yields the required results.

³⁾ $A_{[lm|n]} = A_{lm|n} + A_{mn|l} + A_{nl|m}$. The indices in brackets $\langle \rangle$ are from symmetric and skew symmetric parts.

References

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