

The Lie derivatives in complex areal space

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Abstract. The deformation theories in FINSLER and CARTAN spaces were developed by M. S. KNEBELMAN [1], E. T. DAVIES [2], RUND [3] and YANO [7]. These theories have also been studied in an areal space by TAKANORI IGARASHI [9] and PRASAD [11]. The geometry of spaces in these works are based on real coordinate system. The purpose of this paper is to investigate Lie derivatives in the complex areal space. After giving the outlines of complex areal spaces in § 1 we define the Lie derivative of a vector field in § 2. The section 3 is devoted for rewriting the Lie derivative of a vector field in terms of covariant partial derivatives with respect to z^1, z^{1*} . In § 4 the Lie derivative of the connection coefficients have been obtained. In the last section the concept of the areal motion has been introduced.

Throughout this paper the Latin indices $i, j, h, \dots$ run over 1 to n while Greek indices $\alpha, \beta, \dots$ run over 1 to m .

1. Complex areal spaces.

We consider a $2n$ dimensional real manifold X_{2n} (of class C^∞) referred to local coordinates (x^j, y^j) . Corresponding to each point P of X_{2n} we introduce complex numbers

$$(1.1) \quad z^j = x^j + iy^j \quad (i^2 = -1)$$

which may be regarded as the complex coordinate of P (with respect to given coordinate system). If there exist complex coordinate neighbourhood $U(z^j), U(\bar{z}^j)$ (where $\bar{z}^j$ refer to another local coordinate system) such that in the intersection of these neighbourhoods, we have

$$(1.2) \quad \bar{z}^j = \bar{z}^j(z^h) \quad \det \left\| \frac{\partial \bar{z}^j}{\partial z^h} \right\| \neq 0,$$

where $\bar{z}^j(z^h)$ are holomorphic functions of z^h , then space X_{2n} is said to admit a complex structure. Under these circumstances X_{2n} is called a complex space of (complex) dimension n and is denoted by C_n .

With (1.1) we may associate the conjugate complex

$$(1.3) \quad z^{j*} = x^j - iy^j$$

so that (1.2) carries with it the corresponding conjugate complex transformation

$$(1.4) \quad \bar{z}^{j*} = \bar{z}^{j*}(z^{h*}).$$

An analytic m -dimensional subspace C_m of C_n ($m < n$) is represented parametrically by the equations ([8] page 104)

$$(1.5) \quad z^j = z^j(u^\alpha), \quad z^{j*} = z^{j*}(u^{\alpha*})$$

in which z^j, z^{j*} are holomorphic functions of the complex variables $u^\alpha, u^{\alpha*}$ respectively. Thus the derivatives $\dot{z}_\alpha^j = \frac{\partial z^j}{\partial u^\alpha}$ and their complex conjugate $\dot{z}_{\alpha*}^{j*} = \frac{\partial z^{j*}}{\partial u^{\alpha*}}$ are defined, each of which is an element of an $n \times m$ matrix which is always supposed to be of rank m .

Now we consider real Lagrange function L of the form

$$(1.6) \quad L = L(z^j, z^{j*}, \dot{z}_\alpha^j, \dot{z}_{\alpha*}^{j*})$$

satisfying the conditions

(A) The function L is of class C^4 in all its arguments and it is scalar with respect to the transformations (1.2) and (1.4).

(B) The function L is positive for all independent sets of arguments $\dot{z}_\alpha^j, \dot{z}_{\alpha*}^{j*}$.

(C) The integral

$$(1.7) \quad I = \int_G L du^1 \wedge du^2 \wedge \dots \wedge du^m \wedge du^{1*} \wedge \dots \wedge du^{m*}$$

over a fixed region G of C_m is invariant under the holomorphic transformation of the complex parameters

$$(1.8) \quad \bar{u}^\alpha = \bar{u}^\alpha(u^\beta), \quad \bar{u}^{\alpha*} = \bar{u}^{\alpha*}(u^{\beta*}).$$

(D) The $nm \times nm$ determinant

$$D = \det \left\| m \frac{\partial^2 L^{1/m}}{\partial \dot{z}_\beta^h \partial \dot{z}_{\alpha*}^{j*}} \right\|$$

is non vanishing for linearly independent quantities $\dot{z}_\alpha^h, \dot{z}_{\alpha*}^{j*}$.

The condition C is equivalent to the relation [4]

$$(1.9) \quad (a) \frac{\partial L}{\partial \dot{z}_\alpha^j} \dot{z}_\beta^j = \delta_\beta^\alpha L, \quad (b) \frac{\partial L}{\partial \dot{z}_{\alpha*}^{j*}} \dot{z}_{\beta*}^{j*} = \delta_{\beta*}^{\alpha*} L.$$

In view of (1.9)a and (1.9)b we have ([10])

$$(1.10) \quad 2mL^{1/m} = g_{hj}^{\beta\alpha} \dot{z}_\beta^h \dot{z}_\alpha^j + 2g_{hj*}^{\beta\alpha*} \dot{z}_\beta^h \dot{z}_{\alpha*}^{j*} + g_{h*j*}^{\beta*\alpha*} \dot{z}_{\beta*}^{h*} \dot{z}_{\alpha*}^{j*},$$

where

$$(1.11) \quad g_{hj}^{\beta\alpha}(z^l, z^{l*}, \dot{z}_\lambda^l, \dot{z}_{\lambda*}^{l*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}_\beta^h \partial \dot{z}_\alpha^j},$$

$$(1.12) \quad g_{hj*}^{\beta\alpha*}(z^l, z^{l*}, \dot{z}_\lambda^l, \dot{z}_{\lambda*}^{l*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}_\beta^h \partial \dot{z}_{\alpha*}^{j*}},$$

$$(1.13) \quad g_{h*j*}^{\beta*\alpha*}(z^l, z^{l*}, \dot{z}_\lambda^l, \dot{z}_{\lambda*}^{l*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}_{\beta*}^{h*} \partial \dot{z}_{\alpha*}^{j*}}.$$

From (1.10) it is evident that if L is interpreted as a measure of the area dA of an element of m -dimensional complex subspace ($2m$ dimensional real subspace) spanned by $\dot{z}_\alpha^j, \dot{z}_\alpha^{j*}$ at the points z^1, z^{1*} of C_n in the sense that

$$(1.14) \quad dA = L(z^j, z^{j*}, \dot{z}_\alpha^j, \dot{z}_\alpha^{j*}) du^1 \wedge \dots \wedge du^m \wedge du^{1*} \wedge \dots \wedge du^{m*},$$

then the tensors (1.11), (1.12) and (1.13) can be regarded as a suitable areal metric tensor ([6] page 289). It is to be noted that $g_{hj}^{\beta\alpha}$ is symmetric in pairs of indices such as $(\beta, h), (\alpha, j)$. The similar symmetries exist for tensors $g_{hj}^{\beta\alpha*}$ and $g_{hj}^{\beta\alpha*}$. Furthermore $g_{hj}^{\beta\alpha} \neq g_{jh}^{\beta\alpha}$.

It has been proved in [10] that

$$(1.15) \quad (a) \ g_{hk}^{\beta\gamma} \dot{z}_\beta^h = 0 \quad (b) \ g_{hk}^{\beta\gamma*} \dot{z}_\beta^{h*} = 0.$$

In view of (1.15)a and (1.15)b, the equation (1.10) reduces to

$$(1.16) \quad mL^{1/m} = g_{hj}^{\beta\alpha*} \dot{z}_\beta^h \dot{z}_\alpha^{j*}.$$

The connection coefficients Γ_{kj}^i and Γ_{k*j*}^{i*} defined by ([10])

$$(1.17) \quad \Gamma_{kj}^i = \frac{1}{m} \left[\frac{\partial g_{\beta\alpha}^{im*}}{\partial \dot{z}_\beta^j} \frac{\partial g_{\rho\gamma}^{\alpha*}}{\partial z^k} \dot{z}_\gamma^\rho + g_{\beta\alpha}^{im*} \frac{\partial g_{jm}^{\beta\alpha*}}{\partial z^k} \right],$$

$$(1.18) \quad \Gamma_{k*j*}^{i*} = \frac{1}{m} \left[\frac{\partial g_{\beta\alpha}^{im}}{\partial \dot{z}_\beta^{j*}} \frac{\partial g_{\rho\gamma}^{\alpha*}}{\partial z^{k*}} \dot{z}_\gamma^{\rho*} + g_{\beta\alpha}^{im} \frac{\partial g_{jm}^{\beta\alpha*}}{\partial z^{k*}} \right],$$

are used to define the covariant partial derivatives of a vector field $X_\epsilon^i(z^1, z^{1*}, \dot{z}_\lambda^1, \dot{z}_\lambda^{1*})$ with respect to z^j and z^{j*} . These are given by

$$(1.19) \quad X_{\epsilon|j}^i = \frac{\partial X_\epsilon^i}{\partial z^j} - \frac{\partial X_\epsilon^i}{\partial \dot{z}_\lambda^1} \Gamma_{mj}^l \dot{z}_\lambda^m + \Gamma_{jl}^i X_\epsilon^l,$$

$$(1.20) \quad X_{\epsilon|j*}^{i*} = \frac{\partial X_\epsilon^{i*}}{\partial z^{j*}} - \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_\lambda^{1*}} \Gamma_{m*j*}^{l*} \dot{z}_\lambda^{m*} + \Gamma_{j*l*}^{i*} X_\epsilon^{l*},$$

Similarly the covariant partial derivatives of the vector field $X_\epsilon^{i*}(z^1, z^{1*}, \dot{z}_\lambda^1, \dot{z}_\lambda^{1*})$ is given by

$$(1.21) \quad X_{\epsilon*|j}^{i*} = \frac{\partial X_\epsilon^{i*}}{\partial z^j} - \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_\lambda^1} \Gamma_{mj}^l \dot{z}_\lambda^m,$$

$$(1.22) \quad X_{\epsilon*|j*}^{i*} = \frac{\partial X_\epsilon^{i*}}{\partial z^{j*}} - \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_\lambda^{1*}} \Gamma_{m*j*}^{l*} \dot{z}_\lambda^{m*} + \Gamma_{j*l*}^{i*} X_\epsilon^{l*}.$$

The connection coefficients Γ_{kj}^i and Γ_{k*j*}^{i*} is not in general symmetric in k, j . Therefore, we can also define the second type of covariant partial derivatives of X_ϵ^i and X_ϵ^{i*} with respect to z^j and z^{j*} respectively,

$$(1.23) \quad X_{\epsilon||j}^i = \frac{\partial X_\epsilon^i}{\partial z^j} - \frac{\partial X_\epsilon^i}{\partial \dot{z}_\lambda^1} \Gamma_{mj}^l \dot{z}_\lambda^m + \Gamma_{lj}^i X_\epsilon^l,$$

$$(1.24) \quad X_{\epsilon*||j*}^{i*} = \frac{\partial X_\epsilon^{i*}}{\partial z^{j*}} - \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_\lambda^{1*}} \Gamma_{m*j*}^{l*} \dot{z}_\lambda^{m*} + \Gamma_{l*j*}^{i*} X_\epsilon^{l*}.$$

2. Lie derivatives of a vector field.

Let us consider an infinitesimal point transformation of the form

$$(2.1) \quad (a) \quad \bar{z}^i = z^i + v^i(z^h) dt,$$

$$(2.1) \quad (b) \quad \bar{z}^{i*} = z^{i*} + v^{i*}(z^{h*}) dt,$$

where dt is an infinitesimal constant and v^i, v^{i*} are holomorphic functions of z^h, z^{h*} . The transformations (2.1)a and (2.1)b carries the point (z^i, z^{i*}) of the subspace C_m

$$z^i = z^i(u^a), \quad z^{i*} = z^{i*}(u^{a*})$$

to the neighbouring point $(\bar{z}^i, \bar{z}^{i*})$ of the subspace $\bar{C}_m$

$$\bar{z}^i = \bar{z}^i(u^a), \quad \bar{z}^{i*} = \bar{z}^{i*}(u^{a*}),$$

u^a, u^{a*} being fixed and $v^i(z^h)=0=v^{i*}(z^{h*})$ give the boundary of C_m and $\bar{C}_m$.

Under these transformations the components $\dot{z}_a^i$ and $\dot{z}_{a*}^{i*}$ are deformed as

$$(2.2) \quad (a) \quad \dot{\bar{z}}_a^i = \dot{z}_a^i + \frac{\partial v^i}{\partial z^j} \dot{z}_a^j dt, \quad (b) \quad \dot{\bar{z}}_{a*}^{i*} = \dot{z}_{a*}^{i*} + \frac{\partial v^{i*}}{\partial z^{j*}} \dot{z}_{a*}^{j*} dt,$$

where
$$\dot{z}_a^i = \frac{\partial \bar{z}^i}{\partial u^a}, \quad \dot{z}_{a*}^{i*} = \frac{\partial \bar{z}^{i*}}{\partial u^{a*}}.$$

The variations of $z^i, z^{i*}, \dot{z}_a^i$ and $\dot{z}_{a*}^{i*}$ under (2.1)a and (2.1)b are represented in the form

$$(2.3) \quad (a) \quad \delta z^i = \bar{z}^i - z^i = v^i dt \quad (b) \quad \delta z^{i*} = \bar{z}^{i*} - z^{i*} = v^{i*} dt,$$

$$(2.4) \quad (a) \quad \delta \dot{z}_a^i = \dot{\bar{z}}_a^i - \dot{z}_a^i = \frac{\partial v^i}{\partial z^j} \dot{z}_a^j dt \quad (b) \quad \delta \dot{z}_{a*}^{i*} = \dot{\bar{z}}_{a*}^{i*} - \dot{z}_{a*}^{i*} = \frac{\partial v^{i*}}{\partial z^{j*}} \dot{z}_{a*}^{j*} dt.$$

If a geometric object $\Omega(z^i, z^{i*}, \dot{z}_\lambda^i, \dot{z}_{\lambda*}^{i*})$ is transformed to $\bar{\Omega}(\bar{z}^i, \bar{z}^{i*}, \dot{\bar{z}}_\lambda^i, \dot{\bar{z}}_{\lambda*}^{i*})$ by (2.1)a and (2.1)b then

$$(2.5) \quad \begin{aligned} d\Omega &= \bar{\Omega}(\bar{z}^i, \bar{z}^{i*}, \dot{\bar{z}}_\lambda^i, \dot{\bar{z}}_{\lambda*}^{i*}) - \Omega(z^i, z^{i*}, \dot{z}_\lambda^i, \dot{z}_{\lambda*}^{i*}) = \\ &= \frac{\partial \Omega}{\partial z^i} \delta z^i + \frac{\partial \Omega}{\partial z^{i*}} \delta z^{i*} + \frac{\partial \Omega}{\partial \dot{z}_\lambda^i} \delta \dot{z}_\lambda^i + \frac{\partial \Omega}{\partial \dot{z}_{\lambda*}^{i*}} \delta \dot{z}_{\lambda*}^{i*}. \end{aligned}$$

On the other hand if we interpret (2.1)a and (2.1)b as an infinitesimal coordinate transformation, then neglecting higher order terms with respect to dt , we have

$$(2.6) \quad (a) \quad \frac{\partial \bar{z}^i}{\partial z^i} = \delta_i^i + \frac{\partial v^i}{\partial z^i} dt \quad (b) \quad \frac{\partial \bar{z}^{i*}}{\partial z^{i*}} = \delta_{i*}^{i*} + \frac{\partial v^{i*}}{\partial z^{i*}} dt,$$

$$(2.7) \quad (a) \quad \frac{\partial \bar{z}^i}{\partial \bar{z}^i} = \delta_i^i - \frac{\partial v^i}{\partial \bar{z}^i} dt \quad (b) \quad \frac{\partial \bar{z}^{i*}}{\partial \bar{z}^{i*}} = \delta_{i*}^{i*} - \frac{\partial v^{i*}}{\partial \bar{z}^{i*}} dt.$$

When the geometric object $\Omega(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*})$ is transformed to $\Omega(\bar{z}^I, \bar{z}^{I*}, \dot{\bar{z}}_\lambda^I, \dot{\bar{z}}_{\lambda*}^{I*})$ by the coordinate transformation (2.1)a and (2.1)b then we have

$$(2.8) \quad {}^m d\Omega = \Omega(\bar{z}^I, \bar{z}^{I*}, \dot{\bar{z}}_\lambda^I, \dot{\bar{z}}_{\lambda*}^{I*}) - \Omega(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*}).$$

The Lie derivative of Ω with respect to (v^i, v^{i*}) is defined as ([7], [3])

$$(2.9) \quad \mathfrak{L}_v \Omega = \lim_{dt \rightarrow 0} \frac{{}^v d\Omega - {}^m d\Omega}{dt} = \lim_{dt \rightarrow 0} \frac{\Omega(\bar{z}^I, \bar{z}^{I*}, \dot{\bar{z}}_\lambda^I, \dot{\bar{z}}_{\lambda*}^{I*}) - \Omega(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*})}{dt}.$$

Now let $X_\epsilon^i(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*})$ be a vector field of C_n then using (2.5), (2.3) and (2.4) we have

$$(2.10) \quad {}^v dX_\epsilon^i = \left[\frac{\partial X_\epsilon^i}{\partial z^k} v^k + \frac{\partial X_\epsilon^i}{\partial z^{k*}} v^{k*} + \frac{\partial X_\epsilon^i}{\partial \dot{z}_\lambda^I} \frac{\partial v^I}{\partial z^J} \dot{z}_\lambda^J + \frac{\partial X_\epsilon^i}{\partial \dot{z}_{\lambda*}^{I*}} \frac{\partial v^{I*}}{\partial z^{J*}} \dot{z}_{\lambda*}^{J*} \right] dt.$$

Since the transformation law for the vector field $X_\epsilon^i(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*})$ in C_n is

$$(2.11) \quad X_\epsilon^i(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*}) = \frac{\partial z^i}{\partial \bar{z}^j} X_\epsilon^j(\bar{z}^I, \bar{z}^{I*}, \dot{\bar{z}}_\lambda^I, \dot{\bar{z}}_{\lambda*}^{I*}),$$

therefore, from (2.7)a, (2.8) and (2.11) we have

$$(2.12) \quad {}^m dX_\epsilon^i = X_\epsilon^j \frac{\partial v^i}{\partial z^j} dt.$$

Hence by (2.9)

$$(2.13) \quad \mathfrak{L}_v X_\epsilon^i = \frac{\partial X_\epsilon^i}{\partial z^I} v^I + \frac{\partial X_\epsilon^i}{\partial z^{I*}} v^{I*} + \frac{\partial X_\epsilon^i}{\partial \dot{z}_\lambda^I} \frac{\partial v^I}{\partial z^J} \dot{z}_\lambda^J + \frac{\partial X_\epsilon^i}{\partial \dot{z}_{\lambda*}^{I*}} \frac{\partial v^{I*}}{\partial z^{J*}} \dot{z}_{\lambda*}^{J*} - X_\epsilon^j \frac{\partial v^i}{\partial z^j}.$$

If we consider the vector field $X_\epsilon^{i*}(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*})$ then after using the transformation law

$$(2.14) \quad X_\epsilon^{i*}(z^I, z^{I*}, \dot{z}_\lambda^I, \dot{z}_{\lambda*}^{I*}) = \frac{\partial z^{i*}}{\partial \bar{z}^{j*}} X_\epsilon^{j*}(\bar{z}^I, \bar{z}^{I*}, \dot{\bar{z}}_\lambda^I, \dot{\bar{z}}_{\lambda*}^{I*})$$

for X_ϵ^{i*} we obtain

$$(2.15) \quad \begin{aligned} \mathfrak{L}_v X_\epsilon^{i*} = & \frac{\partial X_\epsilon^{i*}}{\partial z^I} v^I + \frac{\partial X_\epsilon^{i*}}{\partial z^{I*}} v^{I*} + \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_\lambda^I} \frac{\partial v^I}{\partial z^J} \dot{z}_\lambda^J + \\ & + \frac{\partial X_\epsilon^{i*}}{\partial \dot{z}_{\lambda*}^{I*}} \frac{\partial v^{I*}}{\partial z^{J*}} \dot{z}_{\lambda*}^{J*} - X_\epsilon^{j*} \frac{\partial v^{i*}}{\partial z^{j*}}. \end{aligned}$$

In a similar manner, the Lie derivative of covariant vector fields Y_i^ϵ and Y_{i*}^ϵ can be obtained. These are given by

$$(2.16) \quad \begin{aligned} \mathfrak{L}_v Y_i^\epsilon = & \frac{\partial Y_i^\epsilon}{\partial z^I} v^I + \frac{\partial Y_i^\epsilon}{\partial z^{I*}} v^{I*} + \frac{\partial Y_i^\epsilon}{\partial \dot{z}_\lambda^I} \frac{\partial v^I}{\partial z^J} \dot{z}_\lambda^J + \\ & + \frac{\partial Y_i^\epsilon}{\partial \dot{z}_{\lambda*}^{I*}} \frac{\partial v^{I*}}{\partial z^{J*}} \dot{z}_{\lambda*}^{J*} + Y_j^\epsilon \frac{\partial v^j}{\partial z^i}, \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} \mathfrak{L}_v Y_{i^*}^{\varepsilon^*} &= \frac{\partial Y_{i^*}^{\varepsilon^*}}{\partial z^i} v^i + \frac{\partial Y_{i^*}^{\varepsilon^*}}{\partial z^{i^*}} v^{i^*} + \frac{\partial Y_{i^*}^{\varepsilon^*}}{\partial \dot{z}_\lambda^i} \frac{\partial v^i}{\partial z^j} \dot{z}_\lambda^j + \\ &+ \frac{\partial Y_{i^*}^{\varepsilon^*}}{\partial \dot{z}_\lambda^{i^*}} \frac{\partial v^{i^*}}{\partial z^{j^*}} \dot{z}_\lambda^{j^*} + Y_{i^*}^{\varepsilon^*} \frac{\partial v^{j^*}}{\partial z^{i^*}}. \end{aligned}$$

For a scalar $S(z^i, z^{i^*}, \dot{z}_\lambda^i, \dot{z}_\lambda^{i^*})$ we have

$$(2.18) \quad \mathfrak{L}_v S = \frac{\partial S}{\partial z^i} v^i + \frac{\partial S}{\partial z^{i^*}} v^{i^*} + \frac{\partial S}{\partial \dot{z}_\lambda^i} \frac{\partial v^i}{\partial z^j} \dot{z}_\lambda^j + \frac{\partial S}{\partial \dot{z}_\lambda^{i^*}} \frac{\partial v^{i^*}}{\partial z^{j^*}} \dot{z}_\lambda^{j^*}.$$

The theorem (2.1) given below is a direct consequence of equations (2.13), (2.15), (2.16), (2.17), (2.18) and the relation

$$S = X_\varepsilon^i Y_i^\varepsilon = X_{\varepsilon^*}^{i^*} Y_{i^*}^{\varepsilon^*}.$$

Theorem (2.1). *The Lie derivative defined by (2.9) satisfies the Leibnitz rule i.e.*

$$(2.19) \quad \mathfrak{L}_v (X_\varepsilon^i Y_i^\varepsilon) = X_\varepsilon^i (\mathfrak{L}_v Y_i^\varepsilon) + (\mathfrak{L}_v X_\varepsilon^i) Y_i^\varepsilon,$$

$$(2.20) \quad \mathfrak{L}_v (X_{\varepsilon^*}^{i^*} Y_{i^*}^{\varepsilon^*}) = X_{\varepsilon^*}^{i^*} (\mathfrak{L}_v Y_{i^*}^{\varepsilon^*}) + (\mathfrak{L}_v X_{\varepsilon^*}^{i^*}) Y_{i^*}^{\varepsilon^*}.$$

The Lie derivatives of the partial derivative of X_ε^i with respect to $\dot{z}_\alpha^i$ and $\dot{z}_{\alpha^*}^{i^*}$ can be obtained from the first principle and we have

$$(2.21) \quad \begin{aligned} \mathfrak{L}_v \left(\frac{\partial X_\varepsilon^i}{\partial \dot{z}_\alpha^i} \right) &= \frac{\partial^2 X_\varepsilon^i}{\partial z^k \partial \dot{z}_\alpha^i} v^k + \frac{\partial^2 X_\varepsilon^i}{\partial z^{k^*} \partial \dot{z}_\alpha^i} v^{k^*} + \frac{\partial^2 X_\varepsilon^i}{\partial \dot{z}_\beta^k \partial \dot{z}_\alpha^i} \frac{\partial v^k}{\partial z^j} \dot{z}_\beta^j + \\ &+ \frac{\partial^2 X_\varepsilon^i}{\partial \dot{z}_\beta^{k^*} \partial \dot{z}_\alpha^i} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot{z}_\beta^{j^*} - \frac{\partial X_\varepsilon^h}{\partial \dot{z}_\alpha^i} \frac{\partial v^i}{\partial z^h} + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_\alpha^h} \frac{\partial v^h}{\partial z^i} = \\ &= \frac{\partial}{\partial \dot{z}_\alpha^i} \left[\frac{\partial X_\varepsilon^i}{\partial z^k} v^k + \frac{\partial X_\varepsilon^i}{\partial z^{k^*}} v^{k^*} + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_\beta^k} \frac{\partial v^k}{\partial z^j} \dot{z}_\beta^j + \right. \\ &\left. + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_\beta^{k^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot{z}_\beta^{j^*} - X_\varepsilon^h \frac{\partial v^i}{\partial z^h} \right] = \frac{\partial}{\partial \dot{z}_\alpha^i} (\mathfrak{L}_v X_\varepsilon^i), \end{aligned}$$

$$(2.22) \quad \begin{aligned} \mathfrak{L}_v \left(\frac{\partial X_\varepsilon^i}{\partial \dot{z}_{\alpha^*}^{i^*}} \right) &= \frac{\partial^2 X_\varepsilon^i}{\partial z^k \partial \dot{z}_{\alpha^*}^{i^*}} v^k + \frac{\partial^2 X_\varepsilon^i}{\partial z^{k^*} \partial \dot{z}_{\alpha^*}^{i^*}} v^{k^*} + \frac{\partial^2 X_\varepsilon^i}{\partial \dot{z}_\beta^k \partial \dot{z}_{\alpha^*}^{i^*}} \frac{\partial v^k}{\partial z^j} \dot{z}_\beta^j + \\ &+ \frac{\partial^2 X_\varepsilon^i}{\partial \dot{z}_\beta^{k^*} \partial \dot{z}_{\alpha^*}^{i^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot{z}_\beta^{j^*} - \frac{\partial X_\varepsilon^h}{\partial \dot{z}_{\alpha^*}^{i^*}} \frac{\partial v^i}{\partial z^h} + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_{\alpha^*}^h} \frac{\partial v^h}{\partial z^i} = \\ &= \frac{\partial}{\partial \dot{z}_{\alpha^*}^{i^*}} \left[\frac{\partial X_\varepsilon^i}{\partial z^k} v^k + \frac{\partial X_\varepsilon^i}{\partial z^{k^*}} v^{k^*} + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_\beta^k} \frac{\partial v^k}{\partial z^j} \dot{z}_\beta^j + \right. \\ &\left. + \frac{\partial X_\varepsilon^i}{\partial \dot{z}_\beta^{k^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot{z}_\beta^{j^*} - X_\varepsilon^h \frac{\partial v^i}{\partial z^h} \right] = \frac{\partial}{\partial \dot{z}_{\alpha^*}^{i^*}} (\mathfrak{L}_v X_\varepsilon^i). \end{aligned}$$

Hence we have the following:

Theorem (2.2). The operations $\mathcal{L}_v$ and $\frac{\partial}{\partial z_a^i}$ are commutative.

Theorem (2.3). The operations $\mathcal{L}_v$ and $\frac{\partial}{\partial \bar{z}_a^{i*}}$ are commutative.

3. Lie derivative of a vector field in terms of covariant partial derivatives.

Since the transformation vector (v^i, v^{i*}) depends only on z^1 and z^{1*} we have from (1.19), (1.22), (1.23) and (1.24)

$$(3.1) \quad (a) \ v_{|j}^i = \frac{\partial v^i}{\partial z^j} + \Gamma_{jl}^i v^l \quad (b) \ v_{|j*}^{i*} = \frac{\partial v^{i*}}{\partial z^{j*}} + \Gamma_{j*l*}^{i*} v^{l*},$$

$$(3.2) \quad (a) \ v_{||j}^i = \frac{\partial v^i}{\partial z^j} + \Gamma_{lj}^i v^l \quad (b) \ v_{||j*}^{i*} = \frac{\partial v^{i*}}{\partial z^{j*}} + \Gamma_{l*j*}^{i*} v^{l*}.$$

Substituting (1.19), (1.20), (3.1)a, (3.1)b and (3.2)a in (2.13) we get (after some simplification)

$$(3.3) \quad \mathcal{L}_v X_\varepsilon^i = X_{\varepsilon|k}^i v^k + X_{\varepsilon|k*}^{i*} v^{k*} + \frac{\partial X_\varepsilon^i}{\partial \bar{z}_\alpha^i} v_{|j}^l \bar{z}_\alpha^j + \frac{\partial X_\varepsilon^i}{\partial \bar{z}_\alpha^{i*}} v_{|j*}^{l*} \bar{z}_\alpha^{j*} - X_\varepsilon^j v_{||j}^i.$$

Similarly from (2.15), (1.21), (1.22), (3.1)a, (3.1)b and (3.2)b we have

$$(3.4) \quad \mathcal{L}_v X_{\varepsilon*}^{i*} = X_{\varepsilon*|k}^{i*} v^k + X_{\varepsilon*|k*}^{i*} v^{k*} + \frac{\partial X_{\varepsilon*}^{i*}}{\partial \bar{z}_\alpha^i} v_{|j}^l \bar{z}_\alpha^j + \frac{\partial X_{\varepsilon*}^{i*}}{\partial \bar{z}_\alpha^{i*}} v_{|j*}^{l*} \bar{z}_\alpha^{j*} - X_{\varepsilon*}^{j*} v_{||j*}^{i*}.$$

Generalizing this we may express the Lie derivative of an arbitrary tensor

$$T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}$$

of C_n in the form

$$(3.5) \quad \begin{aligned} \mathcal{L}_v T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*} = & v^k T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{|k} + \\ & + v^{k*} T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{|k*} + v_{|j}^i \bar{z}_\alpha^j \frac{\partial}{\partial \bar{z}_\alpha^i} T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*} + \\ & + v_{|j*}^{i*} \bar{z}_\alpha^{j*} \frac{\partial}{\partial \bar{z}_\alpha^{i*}} T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*} - \\ & - \sum_v T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{||v} i_v - \\ & - \sum_\mu T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{||\mu} j_\mu^* + \\ & + \sum_\lambda T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{||\lambda} v_{||\lambda}^k + \\ & + \sum_\theta T_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}{}_{||\theta} v_{||\theta}^{k*}. \end{aligned}$$

In particular the Lie derivative of the metric tensor is given by equations

$$(3.6) \quad \begin{aligned} \mathcal{L}_v g_{ij}^\alpha = & g_{ij|k}^\alpha v^k + g_{ij|k}^\alpha v^{k*} + \frac{\partial g_{ij}^\alpha}{\partial \dot{z}_\lambda^l} v_{|p}^l \dot{z}_\lambda^p + \\ & + \frac{\partial g_{ij}^\alpha}{\partial \dot{z}_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot{z}_{\lambda^*}^{p^*} - g_{kj}^\alpha v_{||i}^k - g_{ik}^\alpha v_{||j}^k, \end{aligned}$$

$$(3.7) \quad \begin{aligned} \mathcal{L}_v g_{ij}^{\alpha*} = & g_{ij|k}^{\alpha*} v^k + g_{ij|k}^{\alpha*} v^{k*} + \frac{\partial g_{ij}^{\alpha*}}{\partial \dot{z}_\lambda^l} v_{|p}^l \dot{z}_\lambda^p + \\ & + \frac{\partial g_{ij}^{\alpha*}}{\partial \dot{z}_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot{z}_{\lambda^*}^{p^*} - g_{kj}^{\alpha*} v_{||i}^k - g_{ik}^{\alpha*} v_{||j}^{k*}, \end{aligned}$$

$$(3.8) \quad \begin{aligned} \mathcal{L}_v g_{i^*j^*}^{\alpha*} = & g_{i^*j^*|k}^{\alpha*} v^k + g_{i^*j^*|k}^{\alpha*} v^{k*} + \frac{\partial g_{i^*j^*}^{\alpha*}}{\partial \dot{z}_\lambda^l} v_{|p}^l \dot{z}_\lambda^p + \\ & + \frac{\partial g_{i^*j^*}^{\alpha*}}{\partial \dot{z}_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot{z}_{\lambda^*}^{p^*} - g_{k^*j^*}^{\alpha*} v_{||i^*}^{k^*} - g_{i^*k^*}^{\alpha*} v_{||j^*}^{k^*}. \end{aligned}$$

4. The Lie derivative of connection coefficient

The Lie derivative of Γ_{kj}^i and $\Gamma_{k^*j^*}^{i*}$ cannot be found directly from (3.5) as these are not components of a tensor. We shall, however, evaluate it from the first principle.

By (2.5) we have

$$(4.1) \quad {}^v d\Gamma_{kj}^i = \left[\frac{\partial \Gamma_{kj}^i}{\partial z^l} v^l + \frac{\partial \Gamma_{kj}^i}{\partial z^{l^*}} v^{l^*} + \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_\alpha^l} \frac{\partial v^l}{\partial z^p} \dot{z}_\alpha^p + \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_{\alpha^*}^{l^*}} \frac{\partial v^{l^*}}{\partial z^{p^*}} \dot{z}_{\alpha^*}^{p^*} \right] dt.$$

The law of transformation of Γ_{kj}^i is given by [10]

$$(4.2) \quad \Gamma_{kj}^i(\bar{z}^l, \bar{z}^{l^*}, \dot{\bar{z}}_\lambda^l, \dot{\bar{z}}_{\lambda^*}^{l^*}) = \frac{\partial \bar{z}^i}{\partial z^p} \left[\frac{\partial^2 z^p}{\partial \bar{z}^k \partial \bar{z}^j} + \Gamma_{\bar{z}^k \bar{z}^j}^{\bar{z}^p} \frac{\partial z^t}{\partial \bar{z}^k} \frac{\partial z^s}{\partial \bar{z}^j} \right].$$

Substituting (2.6)a and (2.7)a in (4.2) we get (after some simplification)

$$(4.3) \quad \begin{aligned} {}^m d\Gamma_{kj}^i = & \Gamma_{kj}^i(\bar{z}^l, \bar{z}^{l^*}, \dot{\bar{z}}_\lambda^l, \dot{\bar{z}}_{\lambda^*}^{l^*}) - \Gamma_{kj}^i(z^l, z^{l^*}, \dot{z}_\lambda^l, \dot{z}_{\lambda^*}^{l^*}) = \\ = & - \left[\frac{\partial^2 v^i}{\partial z^k \partial z^j} - \frac{\partial v^i}{\partial z^l} \Gamma_{kj}^l + \frac{\partial v^l}{\partial z^k} \Gamma_{lj}^i + \frac{\partial v^l}{\partial z^j} \Gamma_{kl}^i \right] dt. \end{aligned}$$

Using the definition of Lie derivative and equations (4.1), (4.3) we have

$$(4.4) \quad \begin{aligned} \mathcal{L}_v \Gamma_{kj}^i = & \left(\frac{\partial \Gamma_{kj}^i}{\partial z^l} - \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_\alpha^h} \Gamma_{pl}^h \dot{z}_\alpha^p \right) v^l + \left(\frac{\partial \Gamma_{kj}^i}{\partial z^{l^*}} - \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_{\alpha^*}^{h^*}} \Gamma_{p^*l^*}^{h^*} \dot{z}_{\alpha^*}^{p^*} \right) v^{l^*} + \\ & + \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_\alpha^h} \left(\frac{\partial v^h}{\partial z^p} + \Gamma_{pl}^h v^l \right) \dot{z}_\alpha^p + \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_{\alpha^*}^{h^*}} \left(\frac{\partial v^{h^*}}{\partial z^{p^*}} + \Gamma_{p^*l^*}^{h^*} v^{l^*} \right) \dot{z}_{\alpha^*}^{p^*} + \\ & + \frac{\partial^2 v^i}{\partial z^k \partial z^j} - \frac{\partial v^i}{\partial z^l} \Gamma_{kj}^l + \frac{\partial v^l}{\partial z^k} \Gamma_{lj}^i + \frac{\partial v^l}{\partial z^j} \Gamma_{kl}^i. \end{aligned}$$

To express the Lie derivative of Γ_{kj}^i in terms of curvature tensor of C_n we consider the expansion of $v_{|k|j}^i$ noting that v^i is independent of $\dot{z}_\lambda^1$ and $\dot{z}_\lambda^{1*}$. Therefore,

$$(4.5) \quad v_{|k|j}^i = v^i \left[\frac{\partial \Gamma_{kl}^i}{\partial z^j} - \frac{\partial \Gamma_{kl}^i}{\partial \dot{z}_\alpha^p} \Gamma_{mj}^p \dot{z}_\alpha^m + \Gamma_{jp}^i \Gamma_{kl}^p \right] + \frac{\partial^2 v^i}{\partial z^j \partial z^k} + \\ + \Gamma_{kl}^i \frac{\partial v^i}{\partial z^j} + \frac{\partial v^i}{\partial z^k} \Gamma_{jl}^i - \frac{\partial v^i}{\partial z^l} \Gamma_{jk}^i - \Gamma_{jk}^p \Gamma_{pl}^i v^i.$$

The curvature tensors R_{kjl}^i and R_{kjl}^{i*} for the connection coefficient Γ_{kj}^i are defined by

$$(4.6) \quad R_{kjl}^i = \left(\frac{\partial \Gamma_{kj}^i}{\partial z^l} - \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_\alpha^h} \Gamma_{pl}^h \dot{z}_\alpha^p \right) - \left(\frac{\partial \Gamma_{kl}^i}{\partial z^j} - \frac{\partial \Gamma_{kl}^i}{\partial \dot{z}_\alpha^h} \Gamma_{pj}^h \dot{z}_\alpha^p \right) + \\ + \Gamma_{kj}^p \Gamma_{pl}^i - \Gamma_{kl}^p \Gamma_{pj}^i,$$

$$(4.7) \quad R_{kjl}^{i*} = \frac{\partial \Gamma_{kj}^{i*}}{\partial z^{l*}} - \frac{\partial \Gamma_{kj}^{i*}}{\partial \dot{z}_{\alpha*}^h} \Gamma_{pl}^{h*} \dot{z}_{\alpha*}^p.$$

A simple calculation based on (4.4), (4.5), (4.6) and (4.7) yields

$$(4.8) \quad \mathcal{L}_v \Gamma_{kj}^i = v_{|k|j}^i + R_{kjl}^i v^l + R_{kjl}^{i*} v^{l*} + \frac{\partial \Gamma_{kj}^i}{\partial \dot{z}_\alpha^h} v_{|p}^h \dot{z}_\alpha^p + \\ + \frac{\partial \Gamma_{kj}^{i*}}{\partial \dot{z}_{\alpha*}^h} v_{|p*}^h \dot{z}_{\alpha*}^p + T_{lj}^i v_{|k}^l + T_{jk}^{i*} v_{|l}^{l*},$$

where

$$(4.9) \quad T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i,$$

is the torsion tensor associated with the connection Γ_{jk}^i .

In a similar manner we can obtain the Lie derivative of the connection coefficients Γ_{k*j*}^{i*} . This is given by

$$(4.10) \quad \mathcal{L}_v \Gamma_{k*j*}^{i*} = v_{|k*|j*}^{i*} + R_{k*j*l*}^{i*} v^{l*} + R_{k*j*l*}^{i*} v^{l*} + \\ + \frac{\partial \Gamma_{k*j*}^{i*}}{\partial \dot{z}_{\alpha*}^h} v_{|p*}^h \dot{z}_{\alpha*}^p + \frac{\partial \Gamma_{k*j*}^{i*}}{\partial \dot{z}_{\alpha*}^h} v_{|p*}^h \dot{z}_{\alpha*}^p + T_{lj*}^{i*} v_{|k*}^{l*} + T_{jk*}^{i*} v_{|l*}^{l*}$$

where

$$(4.11) \quad R_{k*j*l*}^{i*} = \left[\frac{\partial \Gamma_{k*j*}^{i*}}{\partial z^{l*}} - \frac{\partial \Gamma_{k*j*}^{i*}}{\partial \dot{z}_{\alpha*}^h} \Gamma_{pl*}^{h*} \dot{z}_{\alpha*}^p \right] - \left[\frac{\partial \Gamma_{k*l*}^{i*}}{\partial z^{j*}} - \frac{\partial \Gamma_{k*l*}^{i*}}{\partial \dot{z}_{\alpha*}^h} \Gamma_{pj*}^{h*} \dot{z}_{\alpha*}^p \right] + \\ + \Gamma_{k*j*}^{p*} \Gamma_{pl*}^{i*} - \Gamma_{k*l*}^{p*} \Gamma_{pj*}^{i*},$$

$$(4.12) \quad R_{k*j*l}^{i*} = \frac{\partial \Gamma_{k*j*}^{i*}}{\partial z^l} - \frac{\partial \Gamma_{k*j*}^{i*}}{\partial \dot{z}_\alpha^h} \Gamma_{pl}^h \dot{z}_\alpha^p,$$

$$(4.13) \quad T_{jk*}^{i*} = \Gamma_{jk*}^{i*} - \Gamma_{k*j*}^{i*}.$$

5. Areal motion.

In this section we introduce the concept of an areal motion. When the fundamental metric function $L(z^1, z^{1*}, \dot{z}_\lambda^1, \dot{z}_{\lambda*}^{1*})$ satisfies the relation

$$(5.1) \quad \mathcal{L}_v L = 0$$

then the transformations (2.1)a and (2.1)b does not change the area

$$A = \int \int_{(m)} L(z^i, z^{i*}, \dot{z}_\lambda^i, \dot{z}_{\lambda*}^{i*}) du^1 \wedge \dots \wedge du^m \wedge du^{1*} \wedge \dots \wedge du^{m*},$$

of an m -dimensional complex subspace ($2m$ dimensional real subspace). On account of this reason we give the following definition.

Definition (5.1). The transformation given by (2.1)a, (2.1)b is called an areal motion if $\mathcal{L}_v L = 0$.

Theorem (5.1). In order that the space admits an areal motion it is necessary and sufficient that the Lie derivatives of the metric tensor $g_{kj}^{\beta\alpha}$ vanishes.

PROOF. The necessary condition follows from theorems (2.2), (2.3) and the equations (5.1) and (1.12).

The sufficient part follows from theorem (2.1), the equation (1.16) and the facts $\mathcal{L}_v \dot{z}_\beta^h = 0$, $\mathcal{L}_v \dot{z}_{\beta*}^{h*} = 0$.

Theorem (5.2). If the space admits an areal motion then $\mathcal{L}_v g_{hj}^{\beta\alpha} = 0$ and $\mathcal{L}_v g_{h*j*}^{\beta\alpha} = 0$.

The proof of the theorem follows from (1.11), (1.13), (5.1) and the theorems (2.2), (2.3).

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