

On extensions of syntopogenous spaces

By KÁLMÁN MATOLCSY (Debrecen)

Introduction

Theory of *syntopogenous spaces* was worked up in [1]. The present paper will use both the terminology and the notations of this monograph.

For the purpose of the construction of certain *extensions* of syntopogenous spaces, in § 1 the notion of an *inductor* will be defined; this is a monotone mapping (in the sense of [4]) having some additional properties.

In view of its character similar to a *strict extension* of a topological space ([3], ch. 6), in § 2 an extension of a given syntopogenous space will be said to be *tight*, if it can be induced by a special inductor belonging to the trace filters of this extension. The concept of a tight extension can be identified with that of an extension introduced in [6].

§ 3 contains a method to look for a continuous extension of continuous real-valued functions, with the help of which one can get another definition of tight extensions.

Finally, in § 4 we shall consider various conditions on an extension $(E', \mathcal{S}', g)$ of a syntopogenous space $[E, \mathcal{S}]$, under which $g(E)$ is $\mathcal{S}'^b$ -, $\mathcal{S}'^s$ - and $\mathcal{S}'^{sb}$ -dense respectively.

These later conditions characterize also the subspaces of a *double compactification* [5] and of a *completion* [1] of $[E, \mathcal{S}]$, provided a simple separation axiom is satisfied.

The author would like to thank ÁKOS CSÁSZÁR and SÁNDOR GACSÁLYI for their help offered to the shaping of the present form of the paper.

1. Inductors

First of all let us recall the following notions introduced by Á. CSÁSZÁR ([4], (2.1), (2.3), (3.1), (3.19) and (3.20)).

Let E, E' be two sets, $\mathcal{D} \subset 2^E$, finally suppose that $h: \mathcal{D} \rightarrow 2^{E'}$ is a *monotone mapping*, i.e. $D_1, D_2 \in \mathcal{D}$, $D_1 \subset D_2$ imply $h(D_1) \subset h(D_2)$. If $<$ is a semi-topogenous order on E , then

$$h(<) = \bigcup \{<_{h(A), h(B)} : A, B \in \mathcal{D}, A < B\}$$

is a semi-topogenous order on E' (we put $h(<) = <_{\emptyset, E'}$, whenever the family $\{\dots\}$ is empty).

When $\mathcal{A}$ is an order family on E , one can define an order family on E' by the following equality:

$$h(\mathcal{A}) = \{h(<)^\alpha : < \in \mathcal{A}\}.$$

If $\mathfrak{D}$ is a separator for a syntopogenous structure $\mathcal{S}$ on E (i.e. $< \in \mathcal{S}$, $A < B$ imply the existence of a set $D \in \mathfrak{D}$ such that $A \subset D \subset B$), then $h(\mathcal{S})$ is a syntopogenous structure on E' , too.

We prove that any monotone mapping h has an extension H onto 2^E , which preserves certain important properties of the original mapping.

(1.1) Theorem. *If $h: \mathfrak{D} \rightarrow 2^{E'}$ is a monotone mapping, then there exists a monotone mapping $H: 2^E \rightarrow 2^{E'}$ such that*

- (1.1.1) $H|_{\mathfrak{D}} = h$;
- (1.1.2) if $D_1, D_2 \in \mathfrak{D}$ implies $D_1 \cap D_2 \in \mathfrak{D}$ and $h(D_1) \cap h(D_2) = h(D_1 \cap D_2)$, then $H(A) \cap H(B) = H(A \cap B)$ for any $A, B \subset E$;
- (1.1.3) if $\mathfrak{D}$ is a separator for the syntopogenous structure $\mathcal{S}$ on E , then $H(\mathcal{S}) \sim h(\mathcal{S})$;
- (1.1.4) if $s: 2^E \rightarrow 2^{E'}$ is also a monotone mapping such that $s|_{\mathfrak{D}} = h$, then $H(A) \subset s(A)$ for each $A \subset E$.

PROOF. We define the mapping H as follows:

$$H(A) = \{x' \in E' : x' \in h(D), D \subset A \text{ for some } D \in \mathfrak{D}\}.$$

Then it is clear that H is monotone.

- (1.1.1): $D \in \mathfrak{D}$ implies $h(D) \subset H(D)$, and for $x' \in H(D)$ there exists $D_1 \in \mathfrak{D}$ such that $D_1 \subset D$, $x' \in h(D_1)$. Thus we get $x' \in h(D)$, since h is monotone.
- (1.1.2): $A, B \subset E$ obviously implies $H(A \cap B) \subset H(A) \cap H(B)$. Conversely, if $x' \in H(A) \cap H(B)$, then for suitable $D_1, D_2 \in \mathfrak{D}$ we have $D_1 \subset A$, $D_2 \subset B$, $x' \in h(D_1) \cap h(D_2)$. Because of $D_1 \cap D_2 \in \mathfrak{D}$ and $h(D_1) \cap h(D_2) = h(D_1 \cap D_2)$, from $D_1 \cap D_2 \subset A \cap B$ we deduce $x' \in H(A \cap B)$.
- (1.1.3): Obviously $h(\mathcal{S}) \leq H(\mathcal{S})$. Conversely, let $<$ be an element of $\mathcal{S}$, $<_1 \in \mathcal{S}$, $< \subset <_1$. Then $A' H(<) B'$ ($A' \neq \emptyset$, $B' \neq E'$) implies $A < B$, $A' \subset H(A)$ and $H(B) \subset B'$. If $D_1, D_2 \in \mathfrak{D}$ such that $A \subset D_1 <_1 D_2 \subset B$, then $H(A) \subset H(D_1) = h(D_1)$ and $h(D_2) = H(D_2) \subset H(B)$, so that $A' h(<_1) B'$. We got $H(<) \subset h(<_1)$, therefore $H(\mathcal{S}) \leq h(\mathcal{S})$ is clear.
- (1.1.4): From $x' \in H(A)$ the inclusions $x' \in h(D) = s(D) \subset s(A)$ follow for some $A \supset D \in \mathfrak{D}$. ■

In consequence of the theorem, in this paper it will be sufficient to use monotone mappings defined on whole 2^E .

For the sake of further applications, we consider a special class of monotone mappings. Let g be a mapping of a syntopogenous space $[E, \mathcal{S}]$ into a set E' . A monotone mapping $h: 2^E \rightarrow 2^{E'}$ is an *inductor subordinated to $\mathcal{S}$* and g , if the following condition is satisfied:

$$(I_1) < \in \mathcal{S} \text{ and } A < B \text{ imply } g(A) \subset h(B) \text{ and } h(A) \subset E' - g(E - B).$$

(1.2) Theorem. If h is an inductor subordinated to the syntopogenous structure $\mathcal{S}$ on E and to the mapping $g: E \rightarrow E'$, then $g^{-1}(h(\mathcal{S})) \sim \mathcal{S}$. The set $g(E)$ is dense in $[E', h(\mathcal{S})]$, provided the following condition is fulfilled:

(I₂) If $A_j < B_j$ ($1 \leq j \leq n$) for some natural number n and $< \in \mathcal{S}$, then $\bigcap_{j=1}^n B_j = \emptyset$ implies

$$\bigcap_{j=1}^n h(A_j) = \emptyset.$$

PROOF. Let $<$ and $<_1$ be elements of $\mathcal{S}$, $< \subset <_1^3$. Then because of (I₁) $< \subset g^{-1}(h(<_1))$ and $g^{-1}(h(<)) \subset <_1$. Indeed, $A < B$ implies $A <_1 C <_1 D <_1 B$, thus

$$g(A) \subset h(C)h(<_1)h(D) \subset E' - g(E - B),$$

i.e. $Ag^{-1}(h(<_1))B$. Conversely, if $g(A)h(<)E' - g(E - B)$, where $A, B \subset E$, there exists $A_0 < B_0$ such that $g(A) \subset h(A_0)$ and $h(B_0) \subset E' - g(E - B)$. Putting $A_0 <_1 C <_1 D <_1 B_0$, we have $h(A_0) \subset E' - g(E - C)$ and $g(D) \subset h(B_0)$. From this we deduce $g(A) \subset E' - g(E - C)$ and $g(D) \subset E' - g(E - B)$, so that $A \subset C <_1 D \subset B$.

If $< \in \mathcal{S}$ and $x' \in h(<)^q V'$, one can easily prove that

$$x' \in \bigcap_{j=1}^n h(A_j) \quad \text{and} \quad \bigcap_{j=1}^n h(B_j) \subset V',$$

where $A_j < B_j$ ($1 \leq j \leq n$) for some natural number n . Put $<_1 \in \mathcal{S}$, $< \subset <_1^2$ and $A_j <_1 C_j <_1 B_j$ for a suitable $C_j \subset E$ ($1 \leq j \leq n$). With an application of (I₂) we get $\bigcap_{j=1}^n C_j \neq \emptyset$, thus in view of (I₁) $0 \neq \bigcap_{j=1}^n g(C_j) \subset V'$, consequently $g(E) \cap V' \neq \emptyset$. ■

We say that $(E', \mathcal{S}', g)$ is an extension of the syntopogenous space $[E, \mathcal{S}]$, if $[E', \mathcal{S}']$ is a syntopogenous space, and g is an isomorphism of $[E, \mathcal{S}]$ onto a dense subspace of $[E', \mathcal{S}']$. (Cf. [2], p. 238.) We can easily show that $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$, iff g is an injection of E into E' such that $g(E)$ is dense in the syntopogenous space $[E', \mathcal{S}']$ and $g^{-1}(\mathcal{S}') \sim \mathcal{S}$ (cf. [1], (10.14)).

It is also obvious that if $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$, then $(E', \mathcal{S}'^t, g)$ is an extension of $[E, \mathcal{S}^t]$, and $(E', \mathcal{S}'^{tp}, g)$ is that of $[E, \mathcal{S}^{tp}]$.

(1.3) Corollary. Let $[E, \mathcal{S}]$ be a syntopogenous space, g be an injection of E into a set E' , finally let h be an inductor subordinated to $\mathcal{S}$ and g satisfying (I₂). Then $(E', h(\mathcal{S}), g)$ is an extension of $[E, \mathcal{S}]$. ■

In particular, if $g: E \rightarrow E$ is the identity of the set $E = E'$, and h is the inductor subordinated to $\mathcal{S}$ and g defined by $h(A) = A$ for $A \subset E$, then $h(\mathcal{S}) \sim \mathcal{S}$.

2. Tight extensions

Let $[E, \mathcal{S}]$ be a syntopogenous space. A filter base τ in E is round in $[E, \mathcal{S}]$, if for any $R \in \tau$ there exist $R_1 \in \tau$ and $< \in \mathcal{S}$ such that $R_1 < R$ (in this case τ will be called $\mathcal{S}$ -round, too). If τ is an arbitrary filter base in E , then

$$\mathcal{S}(\tau) = \{X \subset E: R < X, < \in \mathcal{S}, R \in \tau\}$$

is an $\mathcal{S}$ -round filter in E , which is called the *neighbourhood filter* of x in $\mathcal{S}$. (We use the more simple notation $\mathcal{S}(x)$ instead of $\mathcal{S}(\{x\})$ for $x \in E$.)

Putting a mapping $g: E \rightarrow E'$ and a filter $\mathfrak{f}'$ in E' , every element of which has a non empty intersection with $g(E)$, by $g^{-1}(\mathfrak{f}')$ we shall mean the filter generated in E by the filter base $\{g^{-1}(F'): F' \in \mathfrak{f}'\}$. If $(E', \mathcal{S}', g)$ is an extension of the syntopogenous space $[E, \mathcal{S}]$, then the filter $\mathcal{S}'(x')$ satisfies the condition mentioned above for any $x' \in E'$, and the filters $g^{-1}(\mathcal{S}'(x'))$ ($x' \in E'$) are called the *trace filters* of this extension.

(2.1) Lemma. *If $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$, then the trace filters are $\mathcal{S}$ -round, and we have*

$$g^{-1}(\mathcal{S}'(g(x))) = \mathcal{S}(x)$$

for $x \in E$. ■

Conversely, let $\mathfrak{f}(x')$ be a filter in E for every $x' \in E'$. Then the equality (cf. [6])

$$h(A) = \{x' \in E' : A \in \mathfrak{f}(x')\} \quad (A \subset E)$$

defines a monotone mapping $h: 2^E \rightarrow 2^{E'}$, which will be called the *monotone mapping belonging to the filters $\mathfrak{f}(x')$* .

The following statement is obvious:

(2.2) Lemma. *If h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$, then $h(\emptyset) = \emptyset$ and $h(A) \cap h(B) = h(A \cap B)$ for $A, B \subset E$, consequently h has property (I_2) , too. ■*

(2.3) Theorem. *Let us suppose that g is a mapping of the syntopogenous space $[E, \mathcal{S}]$ into a set E' , and $\mathfrak{f}(x')$ is a filter in E for any $x' \in E'$. Then the monotone mapping h belonging to these filters is an inductor subordinated to $\mathcal{S}$ and g if, and only if*

$$(2.3.1) \quad \mathcal{S}(\mathfrak{f}(g(x))) = \mathcal{S}(x)$$

holds for any $x \in E$. In this case with the notation $\mathcal{S}' = h(\mathcal{S})$, we have the equality

$$(2.3.2) \quad g^{-1}(\mathcal{S}'(x')) = \mathcal{S}(\mathfrak{f}(x'))$$

for any $x' \in E'$, consequently, if $\mathfrak{f}(x')$ is $\mathcal{S}$ -round, then $g^{-1}(\mathcal{S}'(x'))$ agrees with $\mathfrak{f}(x')$.

PROOF. Suppose that (2.3.1) is satisfied, and put $A < B$, where $< \in \mathcal{S}$. Then $x \in A$ implies $B \in \mathcal{S}(x) \subset \mathfrak{f}(g(x))$, therefore $g(x) \in h(B)$. Let x' be in $h(A)$. If $x' = g(x)$, $x \in E$, then $A \in \mathfrak{f}(g(x))$ implies $B \in \mathcal{S}(x)$, thus $x \in B$. On the basis of this $x' \in E' - g(E - B)$. Conversely, let us assume that (I_1) holds for h . If $B \in \mathcal{S}(x)$, $x \in E$, then $x < A < B$ for an order $< \in \mathcal{S}$, therefore $g(x) \in h(A)$, that is $A \in \mathfrak{f}(g(x))$, thus $B \in \mathcal{S}(\mathfrak{f}(g(x)))$. If $B \in \mathcal{S}(\mathfrak{f}(g(x)))$, then $A < C < B$ for a set $A \in \mathfrak{f}(g(x))$ and a suitable $< \in \mathcal{S}$. $h(A) \subset E' - g(E - C)$ and $g(x) \in h(A)$ imply $x \in C$, so that $B \in \mathcal{S}(x)$.

We prove the equality $g^{-1}(\mathcal{S}'(x')) = \mathcal{S}(\mathfrak{f}(x'))$. Assume $g^{-1}(X') \subset X$, where $x'h(<)X'$ for some $< \in \mathcal{S}$. Then

$$x' \in \bigcap_{j=1}^n h(A_j) \quad \text{and} \quad \bigcap_{j=1}^n h(B_j) \subset X',$$

where $A_j < B_j$ ($1 \leq j \leq n$) for some natural number n . If $<_1 \in \mathcal{S}$, $< \subset <_1^2$ and $A_j <_1 C_j <_1 B_j$ ($1 \leq j \leq n$), then with

$$A = \bigcap_{j=1}^n A_j, \quad C = \bigcap_{j=1}^n C_j \quad \text{and} \quad B = \bigcap_{j=1}^n B_j$$

we have $A \in \mathfrak{f}(x')$ (cf. (2.2)), $A <_1 C$ and $C \subset g^{-1}(h(B)) \subset g^{-1}(X') \subset X$ (cf. (I₁)), thus $X \in \mathcal{S}(\mathfrak{f}(x'))$. On the other hand, if $X \in \mathcal{S}(\mathfrak{f}(x'))$, then $A < X$ for a suitable $A \in \mathfrak{f}(x')$ and $< \in \mathcal{S}$. If $<_1 \in \mathcal{S}$, $< \subset <_1^2$ and $A <_1 C <_1 X$, one can show that $x'h(<_1)h(C)$ and $g^{-1}(h(C)) \subset X$ (cf. (I₁)), therefore $X \in g^{-1}(\mathcal{S}'(x'))$. Finally, if $\mathfrak{f}(x')$ is $\mathcal{S}$ -round, then obviously $\mathcal{S}(\mathfrak{f}(x')) = \mathfrak{f}(x')$. ■

(2.4) Corollary. If g is an injection of the syntopogenous space $[E, \mathcal{S}]$ into a set E' , and $\mathfrak{z}(x')$ is an $\mathcal{S}$ -round filter in E for $x' \in E'$ such that $\mathfrak{z}(g(x)) = \mathcal{S}(x)$ for $x \in E$, then denoting by s the monotone mapping belonging to the filters $\mathfrak{z}(x')$, $(E', s(\mathcal{S}), g)$ is an extension of $[E, \mathcal{S}]$, the trace filters of which are identical with the filters $\mathfrak{z}(x')$. ■

(2.5) Corollary. Under the conditions of (2.4) $(E', s(\mathcal{S})^{tp}, g)$ is a strict extension (see [3], ch. 6) of the topological space $[E, \mathcal{S}^{tp}]$ belonging to the filters $\mathfrak{z}(x')$.

PROOF. As it can be easily seen, $V' \subset E'$ is an $s(\mathcal{S})$ -neighbourhood of a point $x' \in E'$, iff there is a set $V \in \mathfrak{z}(x')$ such that $s(V) \subset V'$. Because of the $\mathcal{S}$ -roundness of $\mathfrak{z}(x')$, the set V can be $\mathcal{S}^{tp}$ -open. ■

In view of (2.5) an extension $(E', \mathcal{S}, g)$ of the syntopogenous space $[E, \mathcal{S}]$ will be called *tight*, if for the monotone mapping s belonging to the trace filters of this extension the equivalence $s(\mathcal{S}) \sim \mathcal{S}'$ is valid. We shall give two characterizations of tight extensions (see (2.10) and (3.7)). The first of these demands the following generalization of the properties of the monotone mapping belonging to a family of filters.

A monotone mapping h will be said to be $(\mathcal{S}, \cap)$ -preserving (or $(\mathcal{S}, \cup)$ -preserving), if $h(\emptyset) = \emptyset$, and for an arbitrary set I of indices, $< \in \mathcal{S}$, $A_i < B_i$ ($i \in I$) imply $\bigcap_{i \in I} h(A_i) \subset h(\bigcap_{i \in I} B_i)$ (or $h(\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} h(B_i)$). If this condition is satisfied by h only for finite sets I of indices, then it will be called *finitely* $(\mathcal{S}, \cap)$ -preserving (or *finitely* $(\mathcal{S}, \cup)$ -preserving).

The finite case can be reduced as follows:

(2.6) Lemma. A monotone mapping h is finitely $(\mathcal{S}, \cap)$ -preserving (or finitely $(\mathcal{S}, \cup)$ -preserving), iff $h(\emptyset) = \emptyset$, and $A_1 < B_1$, $A_2 < B_2$ imply $h(A_1) \cap h(A_2) \subset h(B_1 \cap B_2)$ (or $h(A_1 \cup A_2) \subset h(B_1 \cup B_2)$) for any $< \in \mathcal{S}$.

PROOF. One of the parts of this statement is trivial. We verify the other part with an inductive proof. Let m be a natural number, and let us suppose that $A_j < B_j$ ($1 \leq j \leq n \leq m$) implies $\bigcap_{j=1}^n h(A_j) \subset h(\bigcap_{j=1}^n B_j)$ for any $< \in \mathcal{S}$, finally put $n = m + 1$. If $< \in \mathcal{S}$, $A_j < B_j$ ($1 \leq j \leq n$), then there exists $<_1 \in \mathcal{S}$ such that $< \subset <_1^2$. Assume $A_j <_1 C_j <_1 B_j$ ($1 \leq j \leq n$). In this case

$$\bigcap_{j=1}^m h(A_j) \subset h\left(\bigcap_{j=1}^m C_j\right), \quad \bigcap_{j=1}^m C_j <_1 \bigcap_{j=1}^m B_j$$

and $A_n <_1 B_n$, hence

$$\bigcap_{j=1}^n h(A_j) \subset h\left(\bigcap_{j=1}^m C_j\right) \cap h(A_n) \subset h\left(\bigcap_{j=1}^n B_j\right).$$

The statement concerning the finitely $(\mathcal{S}, \cup)$ -preserving case is totally similar. ■

In our terminology lemma (2.2) can be formulated so that the monotone mapping belonging to a given family of filters is always finitely $(\mathcal{D}_E, \cap)$ -preserving, where $\mathcal{D}_E = \{\subset\}$. But it is obviously finitely $(\mathcal{S}, \cap)$ -preserving for any syntopogenous structure $\mathcal{S}$ on E , and in general we can state:

(2.7) Lemma. *If $\mathcal{S}_1 < \mathcal{S}_2$ and h is a (finitely) $(\mathcal{S}_2, \cap)$ - or $(\mathcal{S}_2, \cup)$ -preserving monotone mapping, then it is (finitely) $(\mathcal{S}_1, \cap)$ - or $(\mathcal{S}_1, \cup)$ -preserving, too. ■*

(2.8) Lemma. *Any finitely $(\mathcal{S}, \cap)$ -preserving monotone mapping has property (I_2) .*

PROOF. In fact, if $< \in \mathcal{S}$, and $A_j < B_j$ ($1 \leq j \leq n$), then $\bigcap_{j=1}^n h(A_j) \subset h\left(\bigcap_{j=1}^n B_j\right) = h(\emptyset) = \emptyset$, provided $\bigcap_{j=1}^n B_j = \emptyset$. ■

(2.9) Lemma. *Let h_1 and h_2 be two monotone mappings. If for any $< \in \mathcal{S}$, $A < B$ implies $h_1(A) \subset h_2(B)$ and $h_2(A) \subset h_1(B)$, then $h_1(\mathcal{S}) \sim h_2(\mathcal{S})$.*

PROOF. Obviously, for $< \subset \subset_1^2$, $<, <_1 \in \mathcal{S}$, we have $h_1(<) \subset h_2(<_1)$ and $h_2(<) \subset h_1(<_1)$, therefore $h_1(\mathcal{S})$ and $h_2(\mathcal{S})$ are equivalent. ■

(2.10) Theorem. *An extension $(E', \mathcal{S}', g)$ of a syntopogenous space $[E, \mathcal{S}]$ is tight, iff there exists a finitely $(\mathcal{S}, \cap)$ -preserving inductor h subordinated to $\mathcal{S}$ and g such that $\mathcal{S}' \sim h(\mathcal{S})$.*

PROOF. If $(E', \mathcal{S}', g)$ is a tight extension, then $\mathcal{S}' \sim s(\mathcal{S})$, where s is the monotone mapping belonging to the trace filters, which is a finitely $(\mathcal{S}, \cap)$ -preserving inductor subordinated to $\mathcal{S}$ and g . After this let h be a finitely $(\mathcal{S}, \cap)$ -preserving inductor subordinated to $\mathcal{S}$ and g such that $\mathcal{S}' \sim h(\mathcal{S})$, and let $\mathfrak{z}(x')$ denote the trace filter of $x' \in E'$. First of all we show that for $X \subset E$: $s(X) \subset h(X)$, and $< \in \mathcal{S}$, $A < B$ imply $h(A) \subset s(B)$. In fact, $x' \in s(X)$ means that $X \in \mathfrak{z}(x')$, and from this $g^{-1}(X') \subset X$, $x' < X'$ follows for some $<' \in \mathcal{S}'$. Let $<$ be an element of $\mathcal{S}$ such that $<' \subset h(<)^q$. Then $A_j < B_j$ ($1 \leq j \leq n$),

$$x' \in \bigcap_{j=1}^n h(A_j), \quad \bigcap_{j=1}^n h(B_j) \subset X'$$

hold for a suitable natural number n . If $<_1 \in \mathcal{S}$, $< \subset \subset_1^2$ and $A_j <_1 C_j <_1 B_j$ ($1 \leq j \leq n$), then using (I_1) , we have

$$\bigcap_{j=1}^n C_j \subset \bigcap_{j=1}^n g^{-1}(h(B_j)) \subset X$$

and

$$x' \in \bigcap_{j=1}^n h(A_j) \subset h\left(\bigcap_{j=1}^n C_j\right) \subset h(X).$$

On the other hand, put $\prec \mathcal{S}$ and $A \prec B$. If $\prec \mathbf{C} \prec_1^2$, $\prec_1 \in \mathcal{S}$, $A \prec_1 C \prec_1 B$ and $\prec' \in \mathcal{S}'$ such that $h(\prec_1)^q \mathbf{C} \prec'$, then $x' \in h(A)$ implies $x' \prec' h(C)$, and because of (I_1) $g^{-1}(h(C)) \subset B$. Thus $B \in \mathfrak{z}(x')$, i.e. $x' \in s(B)$. Finally lemma (2.9) gives that $\mathcal{S}' \sim h(\mathcal{S}) \sim s(\mathcal{S})$, hence the extension in question is tight. ■

3. Extension of functions

Let $\mathbf{R}$ be the real line, $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$, and let $\mathcal{S} = \{\prec_\varepsilon: \varepsilon > 0\}$ be the natural syntopogenous structure on $\mathbf{R}$ defined in [1]. Throughout in this § it will be assumed that $g: E \rightarrow E'$ is a mapping, $\mathcal{S}$ is a syntopogenous structure on E , and $\mathfrak{z}(x')$ is a round filter in $[E, \mathcal{S}]$ for any $x' \in E'$, in particular $\mathfrak{z}(g(x)) = \mathcal{S}(x)$, whenever $x \in E$.

If $f: E \rightarrow \mathbf{R}$ is a real-valued function, we can define a function $f^*: E' \rightarrow \bar{\mathbf{R}}$ by the following formula:

$$f^*(x') = \inf \{ \sup f(A) : A \in \mathfrak{z}(x') \} \quad (x' \in E').$$

(3.1) Lemma. *If $f^*(x') \in \mathbf{R}$, then this is the smallest of all numbers $p \in \mathbf{R}$ such that $f(\mathfrak{z}(x')) \rightarrow p(\mathcal{S})$.*

PROOF. In fact, for a filter base $\mathbf{r}$ in $\mathbf{R}$ the condition $\mathbf{r} \rightarrow p(\mathcal{S})$ is equivalent to the inequality

$$\inf \{ \sup R : R \in \mathbf{r} \} \leq p. \quad \blacksquare$$

(3.2) Lemma. *We have the following properties of f^* :*

(3.2.1) *If f is bounded, then f^* is bounded, too.*

(3.2.2) *$f \leq f^* \circ g$, and if f is $(\mathcal{S}^{tp}, \mathcal{S})$ -continuous, then here equality stands.*

PROOF. (3.2.1) is obvious. (3.2.2): $A \in \mathfrak{z}(g(x)) = \mathcal{S}(x)$ implies $x \in A$, therefore $f(x) \leq \sup f(A)$, and from this we get $f(x) \leq f^*(g(x))$. If f is $(\mathcal{S}^{tp}, \mathcal{S})$ -continuous, then $f(\mathfrak{z}(g(x))) = f(\mathcal{S}(x)) \rightarrow f(x)(\mathcal{S})$, and we have $f^*(g(x)) \leq f(x)$ (cf. (3.1)). ■

(3.3) Lemma. *Let s be the monotone mapping belonging to the filters $\mathfrak{z}(x')$, and suppose that f is bounded. Then we have $f^{*-1}(\prec_\varepsilon) \subset s(f^{-1}(\prec_{\varepsilon/4}))$ and $s(f^{-1}(\prec_\varepsilon)) \subset f^{*-1}(\prec_\varepsilon)$, for each real number $\varepsilon > 0$.*

PROOF. Assume $A' f^{*-1}(\prec_\varepsilon) B'$. This implies the existence of a $p \in \mathbf{R}$ such that $A' \subset f^{*-1}((-\infty, p])$ and $f^{*-1}((-\infty, p+\varepsilon)) \subset B'$. If $A = f^{-1}((-\infty, p+\varepsilon/4])$, $B = f^{-1}((-\infty, p+\varepsilon/2))$, and $x' \in A'$, then $X \subset A$ for some $X \in \mathfrak{z}(x')$, thus $A \in \mathfrak{z}(x')$. If $B \in \mathfrak{z}(x')$, then $f^*(x') \leq p+\varepsilon/2$, therefore $x' \in B'$. We got $A' s(f^{-1}(\prec_{\varepsilon/4})) B'$, because $A f^{-1}(\prec_{\varepsilon/4}) B$.

Conversely, suppose $A' s(f^{-1}(\prec_\varepsilon)) B'$. There exist $A, B \subset E$ such that $A f^{-1}(\prec_\varepsilon) B$, $A' \subset s(A)$ and $s(B) \subset B'$. Let p be the supremum of $f(A)$. $x' \in A'$ implies $A \in \mathfrak{z}(x')$, thus $f^*(x') \leq p$. If for an $x' \in E'$ the inequality $f^*(x') < p+\varepsilon$ holds, then we can take a set $X \subset E$ satisfying the conditions $X \in \mathfrak{z}(x')$ and $\sup f(X) < p+\varepsilon$, thus $X \subset f^{-1}((-\infty, p+\varepsilon)) \subset B$. Consequently $B \in \mathfrak{z}(x')$, therefore $x' \in B'$. This means that $A' f^{*-1}(\prec_\varepsilon) B'$. ■

(3.4) Theorem. Let s be the monotone mapping belonging to the filters $\mathfrak{z}(x')$. Then for any bounded $(\mathcal{S}, \mathcal{I})$ -continuous function f there exists a bounded $(s(\mathcal{S}), \mathcal{I})$ -continuous function f' such that $f = f' \circ g$. ■

(3.5) Corollary. If $(E', \mathcal{S}', g)$ is a tight extension of the syntopogenous space $[E, \mathcal{S}]$, then every bounded $(\mathcal{S}, \mathcal{I})$ -continuous function f has a bounded $(\mathcal{S}', \mathcal{I})$ -continuous extension f' onto E' , i.e. for which $f = f' \circ g$. ■

(3.6) Corollary. If $(E', \mathcal{T}', g)$ is an extension of the topological space $[E, \mathcal{T}]$ such that $\mathcal{T}'$ is a topology, then any bounded $(\mathcal{T}, \mathcal{I})$ -continuous function f has a bounded $(\mathcal{T}', \mathcal{I})$ -continuous extension f' onto E' , i.e. for which $f = f' \circ g$.

PROOF. Let s be the monotone mapping belonging to the trace filters of this extension. Then $s(\mathcal{T})^p$ is coarser than $\mathcal{T}'$ (see (2.5)), therefore every $(s(\mathcal{T})^p, \mathcal{I})$ -continuous function is $(\mathcal{T}', \mathcal{I})$ -continuous, too. ■

Supposing that Φ is an arbitrary functional structure on E (see ch. 12 of [1]), we have a functional structure Φ^* on E' determined as follows:

$$\Phi^* = \{\varphi^* : \varphi \in \Phi\},$$

where

$$\varphi^* = \{f^* : f \in \varphi\}.$$

(3.7) Theorem. Let s denote the monotone mapping belonging to the filters $\mathfrak{z}(x')$. If $\mathcal{S} \sim \mathcal{S}_\Phi$ for an ordering structure Φ on E such that

$$(3.7.1) \quad \varphi_i \in \Phi \ (1 \leq i \leq n) \text{ implies } \bigcup_{i=1}^n \varphi_i \subset \varphi \text{ for some } \varphi \in \Phi,$$

then $s(\mathcal{S}) \sim \mathcal{S}_{\Phi^*}$.

PROOF. If $\{<_i : i \in I\}$ is a family of semi-topogenous orders on E , then

$$(3.7.2) \quad s\left(\bigcup_{i \in I} <_i\right) = \bigcup_{i \in I} s(<_i)$$

Using the terminology of ch. 12 of [1], on the basis of (3.3) and (3.7.2) we can state $<_{\varphi^*, \varepsilon} = \left(\bigcup_{f \in \varphi} f^{*-1}(<_\varepsilon)\right)^q \subset \left(\bigcup_{f \in \varphi} s(f^{-1}(<_{\varepsilon/4}))\right)^q = s\left(\bigcup_{f \in \varphi} f^{-1}(<_{\varepsilon/4})\right)^q = s(<_{\varphi, \varepsilon/4})^q$. From this $\mathcal{S}_{\varphi^*} \leq s(\mathcal{S}_\varphi) \leq s(\mathcal{S})$, therefore $\mathcal{S}_{\varphi^*} \leq s(\mathcal{S})$.

Conversely, $s(<_{\varphi, \varepsilon})^q = s\left(\bigcup_{f \in \varphi} f^{-1}(<_\varepsilon)\right)^q = \left(\bigcup_{f \in \varphi} s(f^{-1}(<_\varepsilon))\right)^q \subset \left(\bigcup_{f \in \varphi} f^{*-1}(<_\varepsilon)\right)^q = <_{\varphi^*, \varepsilon}$, that is $s(\mathcal{S}_\varphi) \leq \mathcal{S}_{\varphi^*}$. From (3.7.1) $\mathcal{S} \sim \mathcal{S}_\Phi \sim \bigcap_{\varphi \in \Phi} \mathcal{S}_\varphi$ follows, thus $s(\mathcal{S}) \sim \bigcap_{\varphi \in \Phi} s(\mathcal{S}_\varphi) = \bigcap_{\varphi \in \Phi} \mathcal{S}_{\varphi^*} \leq \bigcap_{\varphi \in \Phi} \mathcal{S}_{\varphi^*} \leq \bigvee_{\varphi \in \Phi} \mathcal{S}_{\varphi^*} = \mathcal{S}_{\Phi^*}$. ■

(3.7.) shows that if an ordering structure Φ satisfying (3.7.1) compatible with $\mathcal{S}$ is known, then $s(\mathcal{S})$ can be determined without an effective using of s , namely Φ^* is compatible with $s(\mathcal{S})$. This gives a new definition of tight extensions, since such a compatible Φ can be always found for $\mathcal{S}$ (see (12.37) and (12.29) of [1]).

Let us observe that the finiteness of $f^*(x')$ is not guaranteed in the general case, therefore a point $x' \in E'$ will be called *finite from below (from above)*, iff $-\infty < f^*(x') (f^*(x') < +\infty)$ for any $(\mathcal{S}, \mathcal{I})$ -continuous function f on E . Otherwise x' is *infinite from below (or above)*.

For the sake of a characterization of points finite from below or above, let us consider the following notion. A countable system $\mathfrak{B} = \{B_n : n = 0, 1, \dots\}$ will be

said to be *decreasing* in the syntopogenous space $[E, \mathcal{S}]$ (or simply in $\mathcal{S}$), if there is an order $< \in \mathcal{S}$ such that $B_{n+1} < B_n$ for any natural number n . For two systems of sets $\mathfrak{A}$ and $\mathfrak{B}$, we shall use the notation $\mathfrak{A}(\cap)\mathfrak{B} = \{A \cap B : A \in \mathfrak{A}, B \in \mathfrak{B}\}$. Now we can state:

(3.8) Theorem. *A point $x' \in E'$ is finite from below (from above), iff for any decreasing system $\mathfrak{B}$ in $\mathcal{S}$ (in $\mathcal{S}^c$), $\mathfrak{B} \subset \mathfrak{z}(x')$ ($\emptyset \notin \mathfrak{B}(\cap)\mathfrak{z}(x')$) implies $\cap \mathfrak{B} \neq \emptyset$.*

The proof of the theorem is based upon the following lemma:

(3.9) Lemma. *Suppose that $\mathfrak{B}$ is a decreasing system in a syntopogenous space $[E, \mathcal{S}]$, and $\cap \mathfrak{B} = \emptyset$. Then there exists an $(\mathcal{S}, \mathcal{S})$ -continuous function f on E such that for any natural number n , $x \in B_n$ implies $f(x) \leq -n$.*

PROOF. Let $<$ denote an order of $\mathcal{S}$ such that for any natural number n the inequality $B_{n+1} < B_n$ holds. If $\{<_n : n = 0, 1, \dots\} \subset \mathcal{S}$ for which $<_0 = <$ and $<_n \subset <_{n+1}^2$ ($n = 0, 1, \dots$), then for each n there exists a function $t_n : E \rightarrow [0, 1]$ such that

$$(3.9.1) \quad \varepsilon > \frac{1}{2^n} \text{ implies } t_n^{-1}(<_\varepsilon) \subset B_{n+1},$$

$t_n(B_{n+1}) = \{0\}$ and $t_n(E - B_n) = \{1\}$ (see [1], (12.41)). Putting $f_n = t_n - (n+1)$ for any natural number n , f_n also satisfies (3.9.1). We define a function f as follows:

$$f(x) = \begin{cases} 0 & \text{for } x \in E - B_0 \\ f_n(x) & \text{for } x \in B_n - B_{n+1} \end{cases} \quad (x \in E).$$

Because of $E = E - \bigcap_{n=0}^{\infty} B_n = (E - B_0) \cup \left(\bigcup_{n=0}^{\infty} (B_n - B_{n+1}) \right)$, this definition is possible and unambiguous. If $x \in B_n$, then $x \in B_m - B_{m+1}$ for some $m \geq n$, hence $f(x) = f_m(x) = t_m(x) - (m+1) \leq 1 - (m+1) = -m \leq -n$. We show that if r is a real number such that $-(n+1) \leq r < -n$, then

$$(3.9.2) \quad f^{-1}((-\infty, r]) \subset f_n^{-1}((-\infty, r])$$

and

$$(3.9.3) \quad f_n^{-1}((-\infty, r]) \subset f^{-1}((-\infty, r]).$$

In fact, suppose $f(x) \leq r$. If $x \notin B_n$, then $f(x) = f_m(x)$ for some $m < n$, thus $0 = -n + n > r + n \geq f(x) + n \geq f(x) + m + 1 = t_m(x)$, which is impossible by $t_m(E) \subset [0, 1]$, therefore $x \in B_n$. If $x \in B_{n+1}$, then $t_n(x) = 0$, hence $f_n(x) = t_n(x) - (n+1) = -(n+1) \leq r$. If $x \notin B_{n+1}$, then $x \in B_n - B_{n+1}$, that is $f_n(x) = f(x) \leq r$.

Conversely, suppose $f_n(x) < r$. Then $t_n(x) < r + n + 1 < 1$, so that $x \in B_n$. If $x \notin B_{n+1}$, then $x \in B_n - B_{n+1}$, and $f(x) = f_n(x) < r$. If $x \in B_{n+1}$, then for some $k \geq n+1$ we have $f(x) = f_k(x) = t_k(x) - (k+1) \leq 1 - (k+1) = -k \leq -(n+1) \leq r$.

Further we shall verify the $(\mathcal{S}, \mathcal{S})$ -continuity of f . Let ε be a positive real number; without loss of generality we can assume that $\varepsilon < 1$. If $X f^{-1}(<_\varepsilon) Y$, then $X \subset f^{-1}((-\infty, p])$, $f^{-1}((-\infty, p+\varepsilon)) \subset Y$ for a suitable $p \in \mathbb{R}$. If $p \geq 0$, then $Y = E$. Suppose $p < 0$, and let n_0 denote the greatest natural number such that $p < -n_0$. The obviously $-(n_0+1) \leq p$, and $p + \varepsilon < -(n_0-1)$. If $p + \varepsilon/3 < -n_0$, then by (3.9.2)–(3.9.3) $X \subset f_{n_0}^{-1}((-\infty, p])$ and $f_{n_0}^{-1}((-\infty, p+\varepsilon/3)) \subset Y$. If $-n_0 \leq p + \varepsilon/3$, then

$X \subset f_{n_0-1}^{-1}((-\infty, p + \varepsilon/3])$ and $f_{n_0-1}^{-1}((-\infty, p + 2\varepsilon/3)) \subset Y$, and we get

$$X f_{n_0-1}^{-1}(<_{\varepsilon/3}) Y \text{ or } X f_{n_0-1}^{-1}(<_{\varepsilon/3}) Y.$$

Both in the one and in the other case, we have $X <_{m+1} Y$, where m is a natural number such that $\varepsilon/3 > \frac{1}{2^m}$. From this we deduce $f^{-1}(<_{\varepsilon}) \subset <_{m+1}$, and this means that f is $(\mathcal{S}, \mathcal{I})$ -continuous. ■

PROOF OF (3.8). If $x' \in E'$ is infinite from below, then for an $(\mathcal{S}, \mathcal{I})$ -continuous function f on E , $f^*(x') = -\infty$. Then for any natural number n , there is $A \in \mathfrak{z}(x')$ such that $\sup f(A) \leq -n$. In this case $\mathfrak{B} = \{f^{-1}((-\infty, -n]): n=0, 1, \dots\}$ is a decreasing system in $[E, \mathcal{S}]$, $\mathfrak{B} \subset \mathfrak{z}(x')$, and obviously $\bigcap \mathfrak{B} = \emptyset$. Conversely, if $\mathfrak{B} = \{B_n: n=0, 1, \dots\}$ is a decreasing system in $[E, \mathcal{S}]$, for which $\mathfrak{B} \subset \mathfrak{z}(x')$ and $\bigcap \mathfrak{B} = \emptyset$, then $f^*(x') \leq \inf \{\sup f(B_n): n=0, 1, \dots\} = -\infty$, where f is the $(\mathcal{S}, \mathcal{I})$ -continuous function constructed in (3.9).

Suppose that $x' \in E'$ is infinite from above. Then $f^*(x') = +\infty$ for an $(\mathcal{S}, \mathcal{I})$ -continuous function f , and this means $\sup f(A) = +\infty$ for every $A \in \mathfrak{z}(x')$, hence

$$\mathfrak{B} = \{f^{-1}((n, +\infty)): n=0, 1, \dots\}$$

is a decreasing system in $\mathcal{S}^c$ such that $\emptyset \notin \mathfrak{B}(\cap) \mathfrak{z}(x')$ and clearly $\bigcap \mathfrak{B} = \emptyset$. Conversely, let us assume that $\mathfrak{B} = \{B_n: n=0, 1, \dots\}$ is a decreasing system in $\mathcal{S}^c$ such that $\emptyset \notin \mathfrak{B}(\cap) \mathfrak{z}(x')$ and $\bigcap \mathfrak{B} = \emptyset$. Then by (3.9) there exists an $(\mathcal{S}^c, \mathcal{I})$ -continuous function f_0 such that $f_0(x) \leq -n$, whenever $x \in B_n$. In this case $f = -f_0$ is an $(\mathcal{S}, \mathcal{I})$ -continuous function, for which in every $A \in \mathfrak{z}(x')$ a point x lies with $f(x) \geq n$. This shows that $\sup f(A) = +\infty$ for each $A \in \mathfrak{z}(x')$, so that $f^*(x') = +\infty$. ■

(3.10) Theorem. If $\mathcal{S}$ is symmetrical, and $\mathfrak{z}(x')$ is compressed in $\mathcal{S}$, then the point $x' \in E'$ is finite from above, iff it is one from below.

PROOF. Suppose that there is a decreasing system $\mathfrak{B}$ in $\mathcal{S}$ such that $\emptyset \notin \mathfrak{z}(x')(\cap) \mathfrak{B}$ and $\bigcap \mathfrak{B} = \emptyset$. Then denoting by $<$ an order of $\mathcal{S}$ such that $B_{n+1} < B_n$ for any $B_n \in \mathfrak{B}$, we get an other $<_1 \in \mathcal{S}$, for which $< \subset <_1^2$, so that $B_{n+1} <_1 C_n <_1 B_n$ for suitable sets C_n ($n=0, 1, \dots$). $C_n \cap A \neq \emptyset$ for each $A \in \mathfrak{z}(x')$, hence $B_n \in \mathfrak{z}(x')$, that is $\mathfrak{B} \subset \mathfrak{z}(x')$.

On the other hand, if $\mathfrak{B} \subset \mathfrak{z}(x')$ for a decreasing system $\mathfrak{B}$ in $\mathcal{S}$, then $\emptyset \notin \mathfrak{z}(x')(\cap) \mathfrak{B}$ is clear. This shows that the definitions of points finite from below and that of finite from above are equivalent (see (3.8)). ■

(3.11) Theorem. If $\mathcal{S} = \mathcal{T}$ is a topogenous structure, then $x' \in E'$ is finite from below, iff $\mathfrak{z}(x')$ is strongly centred, that is $A_n \in \mathfrak{z}(x')$ ($n=0, 1, \dots$) implies

$$\bigcap_{n=0}^{\infty} A_n \neq \emptyset.$$

PROOF. If $\mathfrak{z}(x')$ is strongly centred, then $\bigcap \mathfrak{B} \neq \emptyset$ for any decreasing system $\mathfrak{B}$ in $\mathcal{T}$ contained in $\mathfrak{z}(x')$, therefore by (3.8) x' is finite from below. Conversely, suppose that $\mathfrak{z}(x')$ is not strongly centred, then there is a countable sequence

$\{A_n: n=0, 1, \dots\} \subset \mathfrak{z}(x')$ with $\bigcap_{n=0}^{\infty} A_n = \emptyset$. We construct a system $\{B_n: n=0, 1, \dots\} \subset \mathfrak{z}(x')$ for which $B_{n+1} < B_n$ ($n=0, 1, \dots$ and $\mathcal{T} = \{<\}$), further $\bigcap_{n=0}^{\infty} B_n = \emptyset$. Assume

that for some fixed natural number n , $0 < m \leq n$ implies the existence of a set $B_m \in \mathfrak{z}(x')$ such that $B_m < B_{m-1}$ and $B_m \subset A_m$, where $B_0 = A_0$. $\mathfrak{z}(x')$ is an $\mathcal{T}$ -round filter, therefore we can find a set $C \in \mathfrak{z}(x')$, for which $C < B_n$. Put $B_{n+1} = C \cap A_{n+1}$. Continuing this procedure, we arrive at the demanded system. By (3.8) this means that x' cannot be finite from below. ■

Applying theorems (3.10) and (3.11) in that case, when $\mathcal{T}$ is the finest symmetrical topogenous structure inducing a given completely regular (or in the terminology of [1] *uniformizable*) topology on E , we get a well-known property of the Čech—Stone compactification, namely in that a point is *finite* in the sense of [3] (finite from below or equivalently from above in our terminology), iff the corresponding trace filter is strongly centred (see ch. 6.4.e of [3]).

4. Conditions of density of $g(E)$ in $[E', \mathcal{S}'^k]$

Let k be an ordinary operator in the sense of [1]. If $(E', \mathcal{S}', g)$ is an extension of the syntopogenous space $[E, \mathcal{S}]$, and $g(E)$ is dense in $[E', \mathcal{S}'^k]$, then $(E', \mathcal{S}'^k, g)$ is obviously an extension of $[E, \mathcal{S}^k]$. In order that this principle be applicable for making syntopogenous extensions (with $\mathcal{S}' = \mathcal{S}'^k$ and $\mathcal{S} = \mathcal{S}^k$), we need to know the conditions of the density of $g(E)$ in $[E', \mathcal{S}'^k]$. We shall consider only the operators $^k = ^b, ^s$ and sb , which are the most particular cases, at the same time they are the most important ones.

Let $(E', \mathcal{S}', g)$ denote a tight extension of $[E, \mathcal{S}]$. For the sake of the formulation of a condition of the density of $g(E)$ in $[E', \mathcal{S}'^b]$, we shall generalize the notion of a Corson filter base of a uniform space (see [3], ch. 8.2.c). A filter base $\mathfrak{r}$ will be said to be *Corson* in the syntopogenous space $[E, \mathcal{S}]$, iff for an arbitrary set I of indices, $< \in \mathcal{S}$, $R_i \in \mathfrak{r}$ and $R_i < B_i$ ($i \in I$) imply $\bigcap_{i \in I} B_i \neq \emptyset$.

Let $\mathfrak{P}(<)$ be, for $< \in \mathcal{S}$, the family of all sets $P \subset E$ such that $A < B$, $P \cap A \neq \emptyset$ imply $P \subset B$ (see [1], p. 220). If $<$ is symmetrical biperfect, and U is the symmetrical reflexive relation associated with that, then $P \in \mathfrak{P}(<)$, iff $(x, y) \in U$ for $x, y \in P$.

(4.1) Lemma. *If for every $< \in \mathcal{S}$ there exists a member of $\mathfrak{P}(<)$, which intersects any element of a filter base $\mathfrak{r}$ in E , then $\mathfrak{r}$ is Corson in $[E, \mathcal{S}]$. The converse is also true, provided $\mathcal{S}$ is a symmetrical syntopology.*

PROOF. If $< \in \mathcal{S}$ and $P \in \mathfrak{P}(<)$, then for any set I of indices, $R_i \in \mathfrak{r}$, $R_i < B_i$, $P \cap R_i \neq \emptyset$ ($i \in I$) imply $\emptyset \neq P \subset \bigcap_{i \in I} B_i$. Conversely, suppose that $\mathcal{S}$ is a symmetrical syntopology, and let $\mathcal{U}$ be the uniformity associated with $\mathcal{S}$. If $\mathfrak{r}$ is Corson in $[E, \mathcal{S}]$, and $U \in \mathcal{U}$ is associated with an arbitrary $< \in \mathcal{S}$, then there exists $U_1 \in \mathcal{U}$ such that $U_1^2 \subset U$. It is obvious that we have a point $x \in \bigcap \{U_1(R) : R \in \mathfrak{r}\}$. Then $P = U_1(x)$ is in $\mathfrak{P}(<)$, since from the symmetricity of U_1 the relation $(y, z) \in U$ follows for any $y, z \in P$. Finally if $R \in \mathfrak{r}$, then there is a point $y \in R$ such that $(y, x) \in U_1$, therefore $y \in P \cap R$. ■

As a simple corollary of (3.8), we can state:

(4.2) **Lemma.** If $\mathfrak{z}(x')$ is a Corson filter in $[E, \mathcal{S}]$, then the point x' is finite from below. ■

(4.3) **Theorem.** Let $(E', \mathcal{S}', g)$ be a tight extension of the syntopogenous space $[E, \mathcal{S}]$ with the trace filter $\mathfrak{z}(x')$ for $x' \in E'$. Then the following statements are equivalent:

(4.3.1) $g(E)$ is dense in $[E', \mathcal{S}'^b]$.

(4.3.2) For every $x' \in E'$, $\mathfrak{z}(x')$ is Corson in $[E, \mathcal{S}]$.

(4.3.3) If $\{f_i: i \in I\}$ is an arbitrary $(\mathcal{S}, \mathcal{S})$ -continuous family of functions on E , then

$$\inf_{x \in E} \sup_{i \in I} f_i(x) \equiv \inf_{x' \in E'} \sup_{i \in I} f_i^*(x').$$

PROOF. (4.3.1) $\Rightarrow$ (4.3.2). Suppose $A_i \in \mathfrak{z}(x')$ and $A_i < B_i$ for a given $< \in \mathcal{S}$ and for an arbitrary set I of indices. Let us consider an order $<' \in \mathcal{S}'$ such that $s(<)^q \subset <'$, where s is the monotone mapping belonging to the trace filters. Then $x' <' s(B_i)$ for $i \in I$, therefore $x' <'^b \bigcap_{i \in I} s(B_i)$. From this we can deduce $\emptyset \neq g^{-1}(\bigcap_{i \in I} s(B_i)) \subset \bigcap_{i \in I} B_i$.

(4.3.2) $\Rightarrow$ (4.3.3). Let us introduce the notations

$$q = \inf_{x \in E} \sup_{i \in I} f_i(x) \quad \text{and} \quad p = \inf_{x' \in E'} \sup_{i \in I} f_i^*(x').$$

Suppose $-\infty < p < +\infty$ and let ε denote an arbitrary positive real number. Then there is a point $x' \in E'$, for which $\sup_{i \in I} f_i^*(x') < p + \varepsilon/2$, consequently a set $A_i \in \mathfrak{z}(x')$ can be found with the property $\sup f(A_i) \leq p + \varepsilon/2$ for every $i \in I$. This implies $A_i \subset f_i^{-1}((-\infty, p + \varepsilon/2]) \subset f_i^{-1}(<_{\varepsilon/2}) \subset f_i^{-1}((-\infty, p + \varepsilon))$. Because of the $(\mathcal{S}, \mathcal{S})$ -continuity of the family of functions in question, one can choose an order $<$ of $\mathcal{S}$ such that $f_i^{-1}(<_{\varepsilon/2}) \subset <$ for each $i \in I$, hence we have

$$A_i < f_i^{-1}((-\infty, p + \varepsilon)) \quad (i \in I).$$

From this we get a point $x \in \bigcap_{i \in I} f_i^{-1}((-\infty, p + \varepsilon))$, for which $\sup_{i \in I} f_i(x) \leq p + \varepsilon$ is obvious. This shows that $q \leq p + \varepsilon$ for any $\varepsilon > 0$, consequently $q \leq p$. If $p = -\infty$, then with a similar train of thought $q = -\infty$ can be deduced. Finally, if $p = +\infty$, the inequality is clear.

(4.3.3) $\Rightarrow$ (4.3.1). Suppose that Φ is an ordering structure compatible with $\mathcal{S}$ (see [1], (12.37)). Then $\mathcal{S}' \sim \mathcal{S}_{\Phi^*}$ (cf. (3.7)), hence for an arbitrary $<' \in \mathcal{S}'$ there exist real numbers $\varepsilon_1, \dots, \varepsilon_n > 0$ and ordering families $\varphi_1, \dots, \varphi_n \in \Phi$ such that

$$\begin{aligned} <' \subset \left(\bigcup_{j=1}^n <_{\varphi_j^*, \varepsilon_j} \right)^q \subset \left(\bigcup_{j=1}^n \left(\bigcup_{f \in \varphi_j} f^{*-1}(<_{\varepsilon_j}) \right)^q \right) = \\ &= \left(\bigcup_{j=1}^n \bigcup_{f \in \varphi_j} f^{*-1}(<_{\varepsilon_j}) \right)^q = \left(\bigcup_{f \in \Phi} f^{*-1}(<_{\varepsilon}) \right)^q, \end{aligned}$$

where $\varepsilon = \min \{\varepsilon_1, \dots, \varepsilon_n\}$ and $\varphi = \bigcup \varphi_j$ (cf. [1], (3.25)). It can be easily seen that $\prec^b \subset (\bigcup_{f \in \varphi} f^{*-1}(\prec_\varepsilon))^{qb} = (\bigcup_{f \in \varphi} f^{*-1}(\prec_\varepsilon^b))^{qb}$ (see [1], (5.22)). Suppose $x' \prec^b V'$ for a point $x' \in E'$, then $x' f_i^{*-1}(\prec_\varepsilon) B'_i$ and $\bigcap_{i \in I} B'_i \subset V'$ holds ($f_i \in \varphi, i \in I$). Since for any constant $c \in \mathbb{R}$ and for any function f , we have $(f+c)^* = f^* + c$, in view of axiom (F_2) of [1], the function f_i can be chosen so that $f_i^*(x') = 0$ ($i \in I$). Then $f_i^{*-1} \cdot ((-\infty, \varepsilon)) \subset B'_i$ for each $i \in I$. As a subfamily of the union of a finite number of $(\mathcal{S}, \mathcal{S})$ -continuous families of functions, the family $\{f_i: i \in I\}$ is also $(\mathcal{S}, \mathcal{S})$ -continuous (see [1], (12.33)). $\sup_{i \in I} f_i^*(x') = 0$, therefore $\inf_{x \in E} \sup_{i \in I} f_i(x) \leq 0$. By (3.2.2) we can state, for a suitable $x \in E$,

$$x \in \bigcap_{i \in I} f_i^{-1}((-\infty, \varepsilon)) = \bigcap_{i \in I} g^{-1}(f_i^{*-1}((-\infty, \varepsilon))) \subset g^{-1}(V'),$$

namely $g(x) \in g(E) \cap V'$. ■

(4.4) Theorem. $(E', \mathcal{S}', g)$ is a tight extension of the syntopogenous space $[E, \mathcal{S}]$ such that $g(E)$ is dense in $[E', \mathcal{S}'^b]$, iff $\mathcal{S}' \sim h(\mathcal{S})$ for a finitely $(\mathcal{S}, \cap)$ -preserving inductor h subordinated to $\mathcal{S}$ and g with the following property:

- (I₃) If $\prec \in \mathcal{S}$, and $A_i \prec B_i$ ($i \in I$) for an arbitrary set I of indices, then $\bigcap_{i \in I} B_i = \emptyset$ implies $\bigcap_{i \in I} h(A_i) = \emptyset$.

PROOF. If $(E', \mathcal{S}', g)$ is a tight extension, then the monotone mapping g satisfies these conditions (see (2.2) and (4.3.2)). Conversely, suppose that $h(\mathcal{S}) \sim \mathcal{S}'$ for a finitely $(\mathcal{S}, \cap)$ -preserving inductor h subordinated to $\mathcal{S}$ and g satisfying (I₃). Then by (2.10) $(E', \mathcal{S}', g)$ is tight. If $\prec' \in \mathcal{S}'$ and $\prec \in \mathcal{S}$ such that $\prec' \subset h(\prec)^q$, then $\prec'^b \subset h(\prec)^{qb} = h(\prec)^b$, therefore $x' \prec^b V'$ implies $x' \in \bigcap_{i \in I} h(A_i)$, $\bigcap_{i \in I} h(B_i) \subset V'$, where $A_i \prec B_i$ ($i \in I$). If $\prec \subset \prec_1^2$, $\prec_1 \in \mathcal{S}$ and $A_i \prec_1 C_i \prec_1 B_i$, then from (I₃) and (I₁) we can deduce

$$\emptyset \neq g(\bigcap_{i \in I} C_i) \subset \bigcap_{i \in I} g(C_i) \subset \bigcap_{i \in I} h(B_i) \subset V',$$

consequently $g(E)$ is dense in $[E', \mathcal{S}'^b]$. ■

Let E, E' be two sets. By the *dual* of a mapping $h: 2^E \rightarrow 2^{E'}$ we shall mean the mapping $h': 2^E \rightarrow 2^{E'}$ determined by the following formula

$$h'(X) = E' - h(E - X) \quad (X \subset E)$$

(cf. [4], (2.12)). If h is monotone, then h' is also monotone. Let $\prec$ be a semi-topogenous order on E , then one can define a semi-topogenous order $h^*(\prec)$ on E' by

$$h^*(\prec) = \bigcup \{h(A), h(B): A \prec B\}.$$

If $\mathfrak{f}(x')$ is a filter in E for each $x' \in E'$, and h is the monotone mapping belonging to these, then for $X \subset E$ we have

$$h'(X) = \{x' \in E': \emptyset \notin \mathfrak{f}(x') \cap \{X\}\}$$

(cf. [5], (4)). In fact,

$$x' \in E' - h'(X) \Leftrightarrow E - X \in \mathfrak{f}(x') \Leftrightarrow \emptyset \in \mathfrak{f}(x')(\cap) \{X\}.$$

In this particular case $h^*(\prec)$ is topogenous, provided $\prec$ is one. If each filter $\mathfrak{f}(x')$ is compressed in the syntopogenous structure $\mathcal{S}$ on E , then

$$h^*(\mathcal{S}) = \{h^*(\prec): \prec \in \mathcal{S}\}$$

is a syntopogenous structure on E' , for which

$$(4.5) \quad h^*(\mathcal{S}) \sim h(\mathcal{S})$$

(cf. [5], (10)).

(4.6) Theorem. Let $(E', \mathcal{S}', g)$ be an extension of the syntopogenous space $[E, \mathcal{S}]$. Then the following statements are equivalent:

(4.6.1) $g(E)$ is dense in $[E', \mathcal{S}']$.

(4.6.2) For each $x' \in E'$ there exists a compressed filter $\mathfrak{f}(x')$ in $[E, \mathcal{S}]$ such that $\mathcal{S}(\mathfrak{f}(g(x))) = \mathcal{S}(x)$ for $x \in E$, and $h^*(\mathcal{S}) \sim \mathcal{S}'$, where h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$.

(4.6.3) $(E', \mathcal{S}', g)$ is a tight extension of $[E, \mathcal{S}]$, and for any $x' \in E'$ there exists a compressed filter $\mathfrak{f}(x')$ in $[E, \mathcal{S}]$ such that the trace filters $\mathfrak{z}(x')$ of the extension agree with the filters $\mathcal{S}(\mathfrak{f}(x'))$ ($x' \in E'$).

(4.6.4) $\mathcal{S}' \sim h(\mathcal{S})$ for a both finitely $(\mathcal{S}, \cap)$ -preserving and finitely $(\mathcal{S}, \cup)$ -preserving inductor h subordinated to $\mathcal{S}$ and g .

(4.6.5) $(E', \mathcal{S}', g)$ is a tight extension of $[E, \mathcal{S}]$, and for any bounded $(\mathcal{S}, \mathcal{I})$ -continuous function f there exists a unique bounded $(\mathcal{S}', \mathcal{I})$ -continuous extension f' onto E' , i.e. for which $f' \circ g = f$.

(4.6.6) $(E', \mathcal{S}', g)$ is a tight extension of $[E, \mathcal{S}]$, and

$$\min \{f_1^*, f_2^*\} = (\min \{f_1, f_2\})^*$$

for any bounded $(\mathcal{S}, \mathcal{I})$ -continuous function f_1, f_2 on E .

Remark. If under condition (4.6.1) $[E', \mathcal{S}']$ is relatively separated with respect to $g(E)$, then it can be embedded into the double compactification of $[E, \mathcal{S}]$ (cf. [5], moreover [1], (16.45)).

PROOF OF (4.6). We shall prove the following implications:

$$(4.6.3) \Rightarrow (4.6.4) \Leftarrow (4.6.6)$$

$$\Uparrow \qquad \Downarrow \qquad \Uparrow$$

$$(4.6.2) \Leftarrow (4.6.1) \Rightarrow (4.6.5)$$

(4.6.1) $\Rightarrow$ (4.6.2). Let $\mathfrak{f}(x')$ be equal to $g^{-1}(\mathcal{S}'^s(x'))$ for each $x' \in E'$. Then $\mathfrak{f}(x')$ is compressed in $[E, \mathcal{S}]$ (cf. [1], (15.45), (15.52) and (15.53)). $g^{-1}(\mathcal{S}'^s) = g^{-1}(\mathcal{S}')^s \sim \mathcal{S}^s$ implies $\mathfrak{f}(g(x)) = \mathcal{S}^s(x)$ for $x \in E$. But $\mathcal{S}(x) \subset \mathcal{S}^s(x)$ is trivial, and owing to $\mathcal{S} \leq \mathcal{S}^2$ we have $\mathcal{S}(x) \subset \mathcal{S}(\mathcal{S}(x)) \subset \mathcal{S}(\mathcal{S}^s(x)) \subset \mathcal{S}(x)$, so that $\mathcal{S}(x) = \mathcal{S}(\mathcal{S}^s(x)) = \mathcal{S}(\mathfrak{f}(g(x)))$.

If $\prec' \in \mathcal{S}'$, $\prec'_1 \in \mathcal{S}'$ such that $\prec' \mathbf{C} \prec'_1^3$, and $\prec \in \mathcal{S}$, for which $g^{-1}(\prec'_1) \mathbf{C} \prec$, then $A' \prec' B'$ and $A' \prec'_1 C' \prec'_1 D' \prec'_1 B'$ imply $A' \subset h(g^{-1}(C'))$ and $h'(g^{-1}(D')) \subset B'$ where h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$. Because of $g^{-1}(C') \prec g^{-1}(D')$ we get $A' h^*(\prec) B'$, thus $\prec' \mathbf{C} h^*(\prec)$.

If $\prec \in \mathcal{S}$, $\prec'$, $\prec'_1 \in \mathcal{S}'$ such that $\prec \mathbf{C} g^{-1}(\prec')$ and $\prec' \mathbf{C} \prec'_1^3$, then $A' h^*(\prec) B'$ implies the existence of sets $A, B \subset E$, for which $A \prec B$, $A' \subset h(A)$ and $h'(B) \subset B'$. If $g(A) \prec'_1 C' \prec'_1 D' \prec'_1 E' - g(E - B)$, then we have $A' \subset C'$ and $D' \subset B'$, hence $A' \prec'_1 B'$. With this $h^*(\prec) \mathbf{C} \prec'_1$.

(4.6.2) $\Rightarrow$ (4.6.3). Let us consider the monotone mapping h belonging to the filters $\mathfrak{f}(x')$. Then by (4.5) $\mathcal{S}' \sim h^*(\mathcal{S}) \sim h(\mathcal{S})$, therefore in view of (2.2), (2.3) and (2.10) $(E', \mathcal{S}', g)$ is a tight extension, the trace filters of which agree with the filters $\mathcal{S}(\mathfrak{f}(x'))$ (see (2.3.2)).

(4.6.3) $\Rightarrow$ (4.6.4). $(E', \mathcal{S}', g)$ is induced by the monotone mapping s belonging to the trace filters $\mathfrak{z}(x')$ of this extension, which is obviously finitely $(\mathcal{S}, \cap)$ -preserving. Let $\mathfrak{f}(x')$ be a compressed filter in $[E, \mathcal{S}]$ for each $x' \in E'$, and put $\mathfrak{z}(x') = \mathcal{S}(\mathfrak{f}(x'))$. If $\prec \in \mathcal{S}$, $A_1 \prec B_1$, $A_2 \prec B_2$, $\prec_1 \in \mathcal{S}$, $\prec \mathbf{C} \prec_1^2$ and $A_i \prec_1 C_i \prec_1 B_i$ ($i=1, 2$), then $A_1 \cup A_2 \in \mathfrak{z}(x')$ implies $C_1 \in \mathfrak{f}(x')$ or $C_2 \in \mathfrak{f}(x')$. From this we get $B_1 \in \mathfrak{z}(x')$ or $B_2 \in \mathfrak{z}(x')$, hence s is finitely $(\mathcal{S}, \cup)$ -preserving by (2.6).

(4.6.4) $\Rightarrow$ (4.6.1). Let h be a both finitely $(\mathcal{S}, \cap)$ - and finitely $(\mathcal{S}, \cup)$ -preserving inductor subordinated to $\mathcal{S}$ and g , further suppose $h(\mathcal{S}) \sim \mathcal{S}'$. If $x' \in E'$, $\prec' \in \mathcal{S}'$ and $x' \prec'' V'$, then there exists $\prec \in \mathcal{S}$ such that $\prec' \mathbf{C} h(\prec)^q$, therefore we have $\prec'' \mathbf{C} h(\prec)^{qs} = h(\prec)^s = \prec''^q$, where $\prec'' = h(\prec) \cup h(\prec)^c$. We can find a natural number n such that $x' \prec'' B'_j$ ($1 \leq j \leq n$), $\bigcap_{j=1}^n B'_j \subset V'$ for some sets $B'_j \subset E'$.

We decompose the set of indices into two sets as follows: $j \in I_1$, iff $x' h(\prec) B'_j$, and $j \in I_2$ otherwise. In this way $j \in I_1$ implies the existence of $A_j, B_j \subset E$, for which $A_j \prec B_j$ and $x' \in h(A_j)$, $h(B_j) \subset B'_j$, moreover $j \in I_2$ implies the existence of $C_j, D_j \subset E$, with $C_j \prec D_j$ and $E' - B'_j \subset h(C_j)$, $h(D_j) \subset E' - x'$ (namely in such a case $x' h(\prec)^c B'_j$). If $\prec_1 \in \mathcal{S}$, $\prec \mathbf{C} \prec_1^2$ and in addition $A_j \prec_1 X_j \prec_1 B_j$ for $j \in I_1$, $C_j \prec_1 Y_j \prec_1 D_j$ for $j \in I_2$, then in view of (I_1) we get the inclusions

$$\bigcap_{j \in I_1} X_j \subset \bigcap_{j \in I_1} g^{-1}(B'_j) \quad \text{and} \quad E - \bigcup_{j \in I_2} Y_j \subset \bigcap_{j \in I_2} g^{-1}(B'_j).$$

We can see that $\emptyset = V' \cap g(E)$ is impossible, because in this case

$$\bigcap_{j \in I_1} X_j \subset \bigcup_{j \in I_2} Y_j,$$

and

$$x' \in \bigcap_{j \in I_1} h(A_j) \subset h\left(\bigcap_{j \in I_1} X_j\right) \subset h\left(\bigcup_{j \in I_2} Y_j\right) \subset \bigcup_{j \in I_2} h(D_j),$$

but this fact contradicts the choice of the sets D_j ($j \in I_2$).

(4.6.1) $\Rightarrow$ (4.6.5). The proof of this implication is based upon the extension theorem (16.45) of [1].

(4.6.5) $\Rightarrow$ (4.6.6). We know that under condition (4.6.5) the $(\mathcal{S}, \mathcal{S})$ -continuity of f_1, f_2 implies the $(\mathcal{S}', \mathcal{S})$ -continuity of f_1^*, f_2^* . We can easily show that both

$\min \{f_1^*, f_2^*\}$ and $(\min \{f_1, f_2\})^*$ are $(\mathcal{S}', \mathcal{I})$ -continuous extensions of the $(\mathcal{S}, \mathcal{I})$ -continuous function $\min \{f_1, f_2\}$, therefore these are equal.

(4.6.6) $\Rightarrow$ (4.6.4). We prove that the monotone mapping s belonging to the trace filters $\mathfrak{z}(x')$ of this extension is finitely $(\mathcal{S}, \cup)$ -preserving. In view of [1], (12.10) and (12.27), we have an ordering structure Φ on E inducing $\mathcal{S}$ such that $\varphi_j \in \Phi$ ($1 \leq j \leq n$) implies the existence of a $\varphi \in \Phi$ fulfilling $\bigcup_{j=0}^n \varphi_j \subset \varphi$. In this case $\mathcal{S} \sim \mathcal{S}_\Phi \sim \bigcup_{\varphi \in \Phi} \mathcal{S}_\varphi$. Put $< \in \mathcal{S}$, $A_1 < B_1$ and $A_2 < B_2$. There exist $\varepsilon > 0$ and $\varphi \in \Phi$ such that $< \subset <_{\varphi, \varepsilon}$. Then $A_j \subset f_j^{-j}((-\infty, 0])$, $f_j^{-j}((-\infty, \varepsilon)) \subset B_j$ for some $f_j \in \varphi$ ($j=1, 2$). If $f_0 = \min \{f_1, f_2\}$ and $x' \in h(A_1 \cup A_2)$, then

$$A_1 \cup A_2 \subset \bigcup_{j=1}^2 f_j^{-1}((-\infty, 0]) = f_0^{-1}((-\infty, 0]),$$

hence $f_0^{-1}((-\infty, 0]) \in \mathfrak{z}(x')$. But in this case, because of the assumption concerning f_0 : $\min \{f_1^*, f_2^*\}(x') = f_0^*(x') \leq 0$. For example put $f_1^*(x') \leq 0$, then there exists $X \in \mathfrak{z}(x')$ such that $\sup f_1(X) < \varepsilon$, thus $X \subset f_1^{-1}((-\infty, \varepsilon)) \subset B_1$, so that $x' \in h(B_1)$.

The proof is complete. ■

In the case of $k=sb$ we have a theorem analogous with (4.6).

(4.7) Theorem. For any extension $(E', \mathcal{S}', g)$ of a syntopogenous space $[E, \mathcal{S}]$ the following statements are equivalent:

- (4.7.1) $g(E)$ is dense in $[E', \mathcal{S}'^{sb}]$.
- (4.7.2) For every $x' \in E'$ there exists a Cauchy filter $\mathfrak{f}(x')$ in $[E, \mathcal{S}]$ such that $\mathcal{S}(\mathfrak{f}(g(x))) = \mathcal{S}(x)$ for $x \in E$, and $h^*(\mathcal{S}) \sim \mathcal{S}'$, where h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$.
- (4.7.3) $(E', \mathcal{S}', g)$ is a tight extension of $[E, \mathcal{S}]$, and for each $x' \in E'$ there exists a Cauchy filter $\mathfrak{f}(x')$ in $[E, \mathcal{S}]$ such that the trace filters of this extension agree with the filters $\mathcal{S}(\mathfrak{f}(x'))$ ($x' \in E'$).
- (4.7.4) $\mathcal{S}' \sim h(\mathcal{S})$ for a both $(\mathcal{S}, \cap)$ - and $(\mathcal{S}, \cup)$ -preserving inductor h subordinated to $\mathcal{S}$ and g . ■

Remark. If under condition (4.7.1) $\mathcal{S}'$ is relatively separated with respect to $g(E)$, then it can be embedded into the completion of $[E, \mathcal{S}]$. This is a consequence of theorem (16.30) of [1].

PROOF. We prove only (3.7.3) $\Rightarrow$ (4.7.4), because the verification of (4.7.1) $\Rightarrow$ $\Rightarrow$ (4.7.2) $\Rightarrow$ (4.7.3) $\Rightarrow$ and (4.7.4) $\Rightarrow$ (4.7.1) is closely similar to that of the corresponding part of (4.6) (4.7.3) $\Rightarrow$ (4.7.4): Let h be the monotone mapping belonging to the filters $\mathfrak{f}(x')$. Since from our conditions $\mathcal{S}(\mathfrak{f}(g(x))) = \mathcal{S}(x)$ follows, h is an inductor subordinated to $\mathcal{S}$ and g by (2.3) We show $\mathcal{S}' \sim h(\mathcal{S})$. In fact, assume that s is the monotone mapping belonging to the trace filters $\mathfrak{z}(x')$ of this extension. We have $s(A) \subset h(A)$ for any $A \subset E$. If $< \in \mathcal{S}$ and $A < B$, then $x' \in h(A)$ implies $B \in \mathcal{S}(\mathfrak{f}(x')) = \mathfrak{z}(x')$, that is $x' \in s(B)$. The extension in question is tight, therefore by (2.9) $\mathcal{S}' \sim s(\mathcal{S}) \sim h(\mathcal{S})$. h is both $(\mathcal{S}, \cap)$ - and $(\mathcal{S}, \cup)$ -preserving. Indeed, $A_i < B_i$

$(i \in I), x' \in \bigcap_{i \in I} h(A_i)$ imply $A_i \in \mathfrak{f}(x')$ for $i \in I$. If $P \in \mathfrak{P}(<) \cap \mathfrak{f}(x')$, then $\emptyset \neq A_i \cap P$ gives $P \subset B_i$ ($i \in I$), thus $\bigcap_{i \in I} B_i \in \mathfrak{f}(x')$, i.e. $x' \in h(\bigcap_{i \in I} B_i)$. If $x' \in h(\bigcap_{i \in I} A_i)$, then $\bigcap_{i \in I} A_i \in \mathfrak{f}(x')$, hence $P \cap A_i \neq \emptyset$ implies $P \subset B_i$ for some $i \in I$ and $P \in \mathfrak{P}(<) \cap \mathfrak{f}(x')$. Thus $B_i \in \mathfrak{f}(x')$ and $x' \in \bigcap_{i \in I} h(B_i)$. ■

References

- [1] Á. CSÁSZÁR, Foundations of General Topology (Oxford—London—New York—Paris) 1963.
- [2] A. CSÁSZÁR, Grundlagen der allgemeinen Topologie (Budapest—Leipzig) 1963.
- [3] Á. CSÁSZÁR, General Topology, (Budapest—Bristol) 1978.
- [4] Á. CSÁSZÁR, Transposition de structures syntopogènes, *Ann. Univ. Budapest, Sect. Math.* 6 (1963), 55—70.
- [5] Á. CSÁSZÁR, Double compactification d'espaces syntopogènes, *Ann. Univ. Budapest., Sect. Math.* 7 (1964), 3—11.
- [6] Á. CSÁSZÁR, Erweiterung, Kompaktifizierung und Vervollständigung syntopogener Räume; Contributions to Extension Theory of Topological Structures *Berlin*, (1969), 51—54.

(Received March 23, 1978.)