

Multipliers of L^1 -algebras with order convolution

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Abstract: For each $n, 1 \leq n \leq N$, let $L^1(a_n, b_n)$ be the commutative convolution measure algebra, under order convolution, of all Lebesgue integrable functions on the interval I_n of real numbers from a_n to b_n , where I_n is a topological semigroup under max multiplication. Then the multiplier algebra $\mathfrak{M}(L^1(a_n, b_n))$ of $L^1(a_n, b_n)$ is shown to be the Banach algebra obtained from $L^1(a_n, b_n)$ by the adjunction of an identity element. Moreover, the Banach space $L^1(\prod_{n=1}^N I_n) = \hat{\otimes}_{n=1}^N L^1(a_n, b_n)$ of Lebesgue integrable functions on the product semigroup $\prod_{n=1}^N I_n$ becomes a commutative convolution measure algebra under order convolution, and it is shown that $\mathfrak{M}(\hat{\otimes}_{n=1}^N L^1(a_n, b_n)) = \hat{\otimes}_{n=1}^N \mathfrak{M}(L^1(a_n, b_n))$.

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1. Introduction

Let N be an arbitrary, but fixed, element of the positive integers $\mathbb{N}$ and, for each $n, 1 \leq n \leq N$, let $I_n = \{a_n, b_n\}$ be an interval of real numbers from a_n to b_n , where either a_n or b_n may possibly be infinite and I_n may or may not contain either a_n or b_n . Then, with the usual interval topology and max multiplication, each I_n is a locally compact idempotent commutative topological semigroup; with the product topology and componentwise multiplication, the product semigroup $S = \prod_{n=1}^N I_n$ has these same properties. Further, let $M(S)$ denote the Banach algebra under convolution product and total variation norm of all finite regular Borel measures on S . Then the Banach space $L^1(S)$ of all measures in $M(S)$ which are absolutely continuous with respect to (product) Lebesgue measure on S becomes a commutative semisimple Banach algebra in the inherited product $*$ which will be called order convolution. A multiplier of $L^1(S)$ is an automatically continuous linear operator $T: L^1(S) \rightarrow L^1(S)$ such that $T(f * g) = f * T(g)$, for all $f, g \in L^1(S)$. In this paper the multipliers of $L^1(S)$ are characterized. The result for the case of a single interval I_n , a special case of which has been obtained by Larsen using different methods [7], is easy to describe: the multiplier algebra $\mathfrak{M}(L^1(I_n))$ of $L^1(I_n)$ is the Banach algebra obtained from $L^1(I_n)$ by adjoining an identity. In the case where $N \geq 2$, the multiplier algebra of $L^1(S)$ is the projective tensor product of the multiplier algebras $\mathfrak{M}(L^1(I_n))$, $1 \leq n \leq N$.

The maximal ideal space $\Delta(L^1(I_n))$ of $L^1(I_n)$, $I_n = \{a_n, b_n\}$, can be identified [6] with the interval $(a_n, b_n]$ (if $b_n = +\infty$, the right-end is compactified). For each $x \in (a_n, b_n]$, if $\varphi_x = K[a_n, x]$ is the characteristic function of $[a_n, x]$, then φ_x

determines a complex homomorphism via the pairing $\hat{f}(x) = \langle \varphi_x, f \rangle = \int_{a_n}^x f(t) dt$, $f \in L^1(I_n)$; correspondingly, $\Delta(L^1(S)) = \prod_{n=1}^N (a_n, b_n]$ [1]. These and other facts about $L^1(I_n) = L^1(a_n, b_n)$ and $L^1(S)$ as Banach algebras can be found in [6, 1]. For results about tensor products, consult [4].

Now, $L^1(S)$ has the structure of a semisimple convolution measure algebra (CMA) in the sense of TAYLOR [8]; hence, there is an isometric imbedding $f \mapsto v_f$ of $L^1(S)$ into $M(\Gamma)$, where Γ is the structure semigroup of $L^1(S)$, and $\Delta(L^1(S)) = \hat{\Gamma}$, the set of continuous semicharacters of Γ . Each $\chi \in \hat{\Gamma}$ determines a unique element of $\Delta(L^1(S))$ via the mapping $f \mapsto \int_{\hat{\Gamma}} \chi dv_f$, $f \in L^1(S)$. Routine verification shows that the structure semigroup Γ_n of $L^1(I_n)$ consists of two ideal elements 0_n and 1_n , and the pairs t^-, t^+ , for all $t, a_n < t < b_n$. Further, $(\Gamma_n, <)$ is linearly ordered by specifying that

$$0_n < s^- < s^+ < t^- < t^+ < 1_n \quad \text{for all } s, t, a_n < s < t < b_n.$$

The product on $\Gamma_n = [0_n, 1_n]$ is then max multiplication, and Γ_n has the compactified order topology (a one or two-point compactification when one of a_n or b_n , or both a_n and b_n , happen to be infinite). Moreover, $\hat{\Gamma}_n = \{\psi_x : x \in (a_n, b_n)\}$, where $\psi_x = K[0, x^+)$, $a_n < x < b_n$, and $\psi_{b_n} = K[0, b_n^+) = K(\Gamma_n)$. Also, because $L^1(S) = \hat{\bigotimes}_{n=1}^N L^1(I_n)$, the projective tensor product of the $L^1(I_n)$, $1 \leq n \leq N$, the structure semigroup Γ of $L^1(S)$ is $\prod_{n=1}^N \Gamma_n$ [11].

The algebra of all multipliers of $L^1(S)$ is denoted by $\mathfrak{M}(L^1(S))$ and each $f \in L^1(S)$ determines a multiplier T_f by $T_f(g) = f * g$, $g \in L^1(S)$. Since $L^1(S)$ has an approximate identity bounded by 1, there is an isometric imbedding $T \mapsto \mu_T$ of $\mathfrak{M}(L^1(S))$ into $M(\Gamma)$ satisfying $v_{T(f)} = \mu_T * v_f$, $f \in L^1(S)$ [5, 12]. This realization of $\mathfrak{M}(L^1(S))$ will be utilized to show that $\mathfrak{M}(L^1(S)) = \hat{\bigotimes}_{n=1}^N \mathfrak{M}(L^1(I_n))$ and that $\mathfrak{M}(L^1(I_n))$ is obtained from $L^1(I_n)$ by adjoining an identity. Section 2 is devoted to the case $N=1$ and contains the answer to a question posed by LARSEN [7, p. 244], while Section 3 is devoted to the general case. We remark that, for CMA's A, B , it is not in general true that $\mathfrak{M}(A \hat{\otimes} B) = \mathfrak{M}(A) \hat{\otimes} \mathfrak{M}(B)$. For example, let $A=B=L^1(\mathbb{R})$; then $\mathfrak{M}(L^1(\mathbb{R})) = M(\mathbb{R})$ and $\mathfrak{M}(L^1(\mathbb{R}) \hat{\otimes} L^1(\mathbb{R})) = \mathfrak{M}(L^1(\mathbb{R} \times \mathbb{R})) = M(\mathbb{R} \times \mathbb{R})$ [10], while $M(\mathbb{R}) \hat{\otimes} M(\mathbb{R}) \subsetneq M(\mathbb{R} \times \mathbb{R})$ [3, pp. 784–5].

2. Multipliers of $L^1(a, b)$

In this section, $N=1$ and we write $L^1(a, b)$ instead of $L^1(S)$. Let $A_f = \{v_f \in M(\Gamma) : f \in L^1(a, b)\}$ and let δ_0 in $M(\Gamma)$ denote the Dirac measure of unit point mass concentrated at the identity 0 of $\Gamma = \Gamma_1$.

Theorem 1. *If $T: L^1(a, b) \rightarrow L^1(a, b)$ is a multiplier of $L^1(a, b)$, then $\mu_T = c\delta_0 + v_f$ for some c in $\mathbb{C}$; f in $L^1(a, b)$. In other words, $\mathfrak{M}(L^1(a, b))$ is the Banach*

algebra obtained by adjoining the identity multiplier to the canonical image of $L^1(a, b)$ in $\mathfrak{M}(L^1(a, b))$.

PROOF. Writing $\mu_T = c\delta_0 + \mu_T^+$, where $c = \mu_T(\{0\})$ (and so $\mu_T^+(\{0\}) = 0$), the result follows provided $\mu_T^+ \in A_\Gamma$. Since A_Γ is norm-closed in $M(\Gamma)$, it suffices to construct a sequence $\{\mu_k\}$ of measures in A_Γ which converges in norm to μ_T^+ . We proceed now to the construction of just such a sequence.

Let $\{c_k\}$ be a strictly decreasing sequence of real numbers in (a, b) converging downward to a , and let $\{u_k\}$ be the approximate identity of $L^1(a, b)$ defined for each k by $u_k = (c_k - c_{k+1})^{-1} K[c_{k+1}, c_k]$. For each k , let $\omega_k = v_{Tu_k}$. Because $v_{Tf} = \mu_T * v_f$, for all $f \in L^1(a, b)$,

$$(1) \quad \langle \psi_x, \mu_T \rangle \langle \psi_x, v_f \rangle = \langle \psi_x, \mu_T * v_f \rangle = \langle \psi_x, v_{Tf} \rangle;$$

that is,

$$(2) \quad \mu_T[0, x^+) \int_a^x f(t) dt = v_{Tf}[0, x^+), \quad x \in (a, b], f \in L^1(a, b).$$

Setting $f = u_k$ in (2) and observing that $\int_a^x u_k(t) dt = 1$, for all x in $[c_k, b]$, yields

$$(3) \quad \mu_T[0, x^+) = \omega_k[0, x^+), \quad x \in [c_k, b].$$

Further, since μ_T and ω_k are regular Borel measures on Γ , for all x in $(c_k, b]$,

$$(4) \quad \mu_T[c_k^+, x^+) = \omega_k[c_k^+, x^+),$$

with the convention that $[c_k^+, b^+) = [c_k^+, 1]$.

Now, for k in $\mathbb{N}$, define the measures μ_k and λ_k on Γ by setting $\mu_k = K[c_k^+, 1]\mu_T$ and $\lambda_k = K[c_k^+, 1]\omega_k$. Then, for each $\psi_x \in \hat{\Gamma}$, $x \in (a, b]$,

$$(5) \quad \begin{aligned} \langle \psi_x, \mu_k \rangle &= \int_{\Gamma} \psi_x(s) d\mu_k(s) = \int_{\Gamma} K[0, x^+)(s) K[c_k^+, 1](s) d\mu_T(s) = \\ &= \begin{cases} 0, & \text{if } a < x \leq c_k \\ \mu_T[c_k^+, x^+), & \text{if } c_k < x \leq b, \end{cases} \end{aligned}$$

and, in like manner,

$$(6) \quad \langle \psi_x, \lambda_k \rangle = \begin{cases} 0, & \text{if } a < x \leq c_k \\ \omega_k[c_k^+, x^+), & \text{if } c_k < x \leq b. \end{cases}$$

Thus, in view of the identity (4), the measures μ_k and λ_k agree on $\hat{\Gamma}$; hence, $\mu_k = \lambda_k$. Consequently, $\mu_k \ll \omega_k$ and, since $\omega_k \in A_\Gamma$ and A_Γ is an L -subspace of $M(\Gamma)$, $\mu_k \in A_\Gamma$, for every k in $\mathbb{N}$.

The proof will be completed by setting $c = \mu_T(\{0\})$ and $\mu_T^+ = \mu_T - c\delta_0$, and proving that $\mu_T^+ = \lim_{k \rightarrow \infty} \mu_k$ in norm. Toward this end, observe that

$$(7) \quad \mu_T^+ - \mu_k = K(0, 1]\mu_T - K[c_k^+, 1]\mu_T = K(0, c_k^+)\mu_T,$$

so

$$(8) \quad \|\mu_T^+ - \mu_k\| = |K(0, c_k^+)\mu_T|(\Gamma) = |\mu_T|(0, c_k^+),$$

for each k . However, the interval $(0, c_1^+)$ is the countable disjoint union $\bigcup_{n=1}^{\infty} [c_{k+1}^+, c_k^+)$ and $|\mu_T|$ is a finite regular Borel measure on Γ , so

$$(9) \quad \sum_{k=1}^{\infty} |\mu_T|[c_{k+1}^+, c_k^+) = |\mu_T|(0, c_1^+) < +\infty.$$

Therefore, for every $\varepsilon > 0$, there exists an integer K such that

$$(10) \quad |\mu_T|(0, c_K^+) = \sum_{k=K}^{\infty} |\mu_T|[c_{k+1}^+, c_k^+) < \varepsilon.$$

This fact, together with equation (8), yields $\|\mu_T^+ - \mu_k\| < \varepsilon$, for all $k \geq K$. ■

LARSEN [7] obtained Theorem 1 for the special case $S=[0, 1]$ (i.e., $L^1(0, 1)$) using methods quite different from ours. Larsen also observed that, if a multiplier T of $L^1(0, 1)$ is a compact operator on $L^1(0, 1)$, then $T=T_f$, for some f in $L^1(0, 1)$. He stated the converse of this result as an open question, to which we now supply the answer.

Proposition 2. *For each positive integer n , the multiplier T_n of $L^1(0, 1)$ defined by the polynomial nx^{n-1} (i.e., $T_n f = (nx^{n-1}) * f$, for all f in $L^1(0, 1)$) is neither a compact nor even a weakly compact operator on $L^1(0, 1)$.*

PROOF. First, we exhibit a sequence $\{f_m\}_m$, contained in the unit ball of $L^1(0, 1)$ such that, for no n , does the sequence $\{T_n f_m\}_m$ have a norm-convergent subsequence. Let $f_1 \equiv 1$ and, for $m \geq 2$, let $f_m = f_{m-1} * f_1$; then [6, MIDDLE, p. 4] $f_m(x) = mx^{m-1}$, for each m . Observe that $T_n f_m = f_n * f_m = f_{n+m}$, for all n, m in $\mathbb{N}$.

Now, for each m , $f_m \geq 0$ and $\|f_m\|_1 = \int_0^1 mx^{m-1} dx = 1$. Moreover, since $\lim_{m \rightarrow \infty} f_m(x) = 0$, for all x in $[0, 1)$, the only possible limit function in $L^1(0, 1)$ for a norm-convergent subsequence of the sequence $\{T_n f_m\}_m = \{f_{n+m}\}_m$ is the zero function. However, $\|f_{n+m}\|_1 = 1$, for all m , so every member of the sequence $\{T_n f_m\}_m$ is L^1 -distance 1 from the zero function. Therefore, the norm-bounded sequence $\{f_m\}_m$ in $L^1(0, 1)$ is such that, for no n , does the sequence $\{T_n f_m\}_m$ possess a norm-convergent subsequence. Hence, the multiplier T_n of $L^1(0, 1)$ is not a compact operator on $L^1(0, 1)$, for any n .

Next, a routine calculation shows that $T_n^2 = T_{2n}$, for all n ; hence, the operator T_n^2 is not compact, for any n . However, the product of two weakly compact operators on an L^1 -space is compact [2, Cor. 13, p. 510]. Thus, the multiplier T_n is not weakly compact, for any n . ■

3. Multipliers of $L^1(\Pi\{a_n, b_n\})$

Let S be the product semigroup $\prod_{n=1}^N I_n$, $N \geq 2$. If $\{u_{kn}\}_k$ is the approximate identity for $L^1(I_n)$ constructed in the proof of Theorem 1 from a strictly decreasing sequence $\{c_{kn}\}_k$ of real numbers in (a_n, b_n) then $\{u_k = \bigotimes_{n=1}^N u_{kn}\}_k$ is an approximate

identity bounded by one for $L^1(S)$ which consists of functions whose Gelfand transforms have compact support. This fact, together with the regularity of $L^1(S)$, implies that $\mathfrak{M}(L^1(S))$ is a CMA and that its canonical image in $M(\Gamma)$ is an L -subspace of $M(\Gamma)$ [12]. Moreover, $\bigotimes_{n=1}^N \mathfrak{M}(L^1(I_n))$ is imbedded as an L -subalgebra in $\mathfrak{M}\left(\bigotimes_{n=1}^N L^1(I_n)\right) = \mathfrak{M}(L^1(S))$ [8, Prop. 2.5.2].

Now, for each n , $1 \leq n \leq N$, Γ_n denotes the structure semigroup of $L^1(I_n)$, and $v^{(n)}: L^1(I_n) \rightarrow M(\Gamma_n)$ denotes the canonical isometric imbedding of $L^1(I_n)$ in $M(\Gamma_n)$. Further, if $A_{\Gamma_n} = \{v_{f_n}^{(n)} \in M(\Gamma_n): f_n \in L^1(I_n)\}$, then the map $(f_n) \mapsto (v_{f_n}^{(n)})$ defines an isometric isomorphism from $\prod_{n=1}^N L^1(I_n)$ onto $\prod_{n=1}^N A_{\Gamma_n}$. It is readily established that $\bigotimes v^{(n)}$ is an isometric isomorphism from the dense subspace $\bigotimes_{n=1}^N L^1(I_n)$ of $L^1(S)$ onto the dense subspace $\bigotimes_{n=1}^N A_{\Gamma_n}$ of $\bigotimes_{n=1}^N A_{\Gamma_n} = A_\Gamma = \{v_f \in M(\Gamma): f \in L^1(S)\}$, and that $\bigotimes v^{(n)}$ has a unique continuous extension to an isometric isomorphism $v = \bigotimes v^{(n)}$ from $L^1(S)$ onto A_Γ .

The following notation will simplify the statement and proof of the upcoming theorem. For $1 \leq m \leq N$, let $\Gamma'_m = \{t = (t_n) \in \Gamma: t_m = 0_m\}$ and let $I'_m = \prod_{n \neq m} I_n$. Further, for each m , let id_m denote the identity operator on $L^1(a_m, b_m)$. Finally, if $\mu \in M(\Gamma)$, then $\text{supp}(\mu)$ will denote its support.

Theorem 3. *If $T: L^1(S) \rightarrow L^1(S)$ is a multiplier of $L^1(S)$ and μ_T is the measure in $M(\Gamma)$ corresponding to T , then μ_T admits the decomposition:*

$$\mu_T = v_f + \sum_{m=1}^N \tau_m,$$

where $f \in L^1(S)$ (i.e., $v_f \in A_\Gamma$) and, for $1 \leq m \leq N$, τ_m is a measure in $M(\Gamma)$ with $\text{supp}(\tau_m) \subseteq \Gamma'_m$ and corresponds to a multiplier T_m of $L^1(S)$ contained in $\mathbb{C} \cdot \text{id}_m \otimes \mathfrak{M}(L^1(I'_m))$. Therefore, $\mathfrak{M}(L^1(S)) = \bigotimes_{n=1}^N \mathfrak{M}(L^1(a_n, b_n))$.

PROOF. The last statement follows from the first by using induction on N . To prove the first statement of the theorem, fix T in $\mathfrak{M}(L^1(S))$ and let μ_T be the corresponding measure in $M(\Gamma)$. Let $F = \bigcup_{m=1}^N \Gamma'_m$ (the union of the faces) and define two new measures μ_T^+ , μ_T^0 on Γ by setting $\mu_T^0 = K(F)\mu_T$ and $\mu_T^+ = \mu_T - \mu_T^0$. We will show that $\mu_T^+ \in A_\Gamma$ (hence, $\mu_T^+ = v_f$, for some f in $L^1(S)$), and that $\mu_T^0 = \sum_{m=1}^N \tau_m$, with τ_m as in the statement of the theorem. We begin with μ_T^+ .

For each k in $\mathbb{N}$, let $\omega_k = v_{T\omega_k} \in A_\Gamma$. Now since $v_{Tf} = \mu_T * v_f$, for all f in $L^1(S)$, it follows that, for each ψ_x in $\hat{F}$, where $x = (x_n) \in \prod(a_n, b_n]$,

$$(11) \quad \langle \psi_x, \mu_T \rangle \langle \psi_x, v_f \rangle = \langle \psi_x, \mu_T * v_f \rangle = \langle \psi_x, v_{Tf} \rangle;$$

that is

$$(12) \quad \mu_T(\Pi[0_n, x_n^+])f(x) = v_{Tf}(\Pi[0, x_n^+]).$$

Setting $f = u_k = \otimes u_{kn}$ in equation (12) yields the identity:

$$(13) \quad \mu_T(\Pi[0_n, x_n^+]) = \omega_k(\Pi[0_n, x_n^+]),$$

valid for all (x_n) in $\Pi[c_{kn}, b_n]$. Using equation (13) and the fact that μ_T and ω_k are regular Borel measures on Γ , a straightforward but tedious computation shows that

$$(14) \quad \mu_T(\Pi[c_{kn}^+, x_n^+]) = \omega_k(\Pi[c_{kn}^+, x_n^+]),$$

for all (x_n) in $\Pi(c_{kn}, b_n]$. The computation is indicated for $N=2$ only; the general case is similar. For $N=2$, the set $\prod_{n=1}^2 [0_n, x_n^+)$ may be written as a disjoint union of sets as follows:

$$(15) \quad [O_1, x_1^+) \times [O_2, x_2^+) = [O_1, c_{k1}^+) \times [O_2, c_{k2}^+) \cup \\ \cup [O_1, c_{k1}^+) \times [c_{k2}^+, x_2^+) \cup [c_{k1}^+, x_1^+) \times [O_2, c_{k2}^+) \cup [c_{k1}^+, x_1^+) \times [c_{k2}^+, x_2^+).$$

By (13), μ_T and ω_k agree on all of the sets in equation (15), except the final set; hence, they agree on the final set as well, establishing equation (14). For general N , the set $\prod_{n=1}^N [O_n, x_n^+)$ decomposes into 2^N disjoint sets, one of which is $\prod_{n=1}^N [c_{kn}^+, x_n^+)$. Since equation (13) implies that μ_T and ω_k agree on $\Pi[O_n, x_n^+)$ and on all of the other $2^N - 1$ sets in the decomposition of $\Pi[O_n, x_n^+)$, the identity (14) follows.

Next, for k in N , set $R_k = \prod_{n=1}^N [c_{kn}^+, 1_n] \subseteq \Gamma$ and define measures $\mu_k = K(R_k)\mu_T$ and $\lambda_k = K(R_k)\omega_k$ on Γ . Then, for each $\psi_x \in \hat{f}$, $x = (x_n) \in \Pi(a_n, b_n]$,

$$(16) \quad \langle \psi_x, \mu_k \rangle = \int_{\Gamma} \psi_x d\mu_k = \int_{\Gamma} K(\Pi[O_n, x_n^+]) K(R_k) d\mu_T =$$

$$= \begin{cases} 0, & \text{if } x_n \leq c_{kn}, \text{ for some } n \\ \mu_T(\Pi[c_{kn}^+, x_n^+]), & \text{if } (x_n) \in \Pi(c_{kn}^+, b_n], \end{cases}$$

and, analogously,

$$(17) \quad \langle \psi_x, \lambda_k \rangle = \begin{cases} 0, & \text{if } x_n \leq c_{kn}, \text{ for some } n \\ \omega_k(\Pi[c_{kn}^+, x_n^+]), & \text{if } (x_n) \in \Pi(c_{kn}^+, b_n]. \end{cases}$$

Therefore, by equation (14), μ_k and λ_k agree on $\hat{f}$ and, as a result $\mu_k = \lambda_k$, implying that $\mu_k \in A_{\Gamma}$, for every k .

Now, because A_{Γ} is a norm-closed subspace of $M(\Gamma)$, $\mu_T^+ \in A_{\Gamma}$ provided the sequence $\{\mu_k\}_k$ of measures in A_{Γ} converges in norm to μ_T^+ . To see that this

occurs, first observe that $\mu_T^\pm - \mu_k = \mu_T - K(F)\mu_T - K(R_k)\mu_T = K((\Gamma \setminus F) \setminus R_k)\mu_T$, so

$$(18) \quad \|\mu_T^\pm - \mu_k\| = |\mu_T^\pm - \mu_k|(\Gamma) = |\mu_T|((\Gamma \setminus F) \setminus R_k).$$

However, the product interval $\Gamma \setminus F = \prod_{n=1}^N (0_n, 1_n]$ is the countable disjoint union $\bigcup_{k=0}^{\infty} (R_{k+1} \setminus R_k)$, when R_0 is taken to be the empty set. Hence, because $|\mu_T|$ is a finite regular Borel measure on Γ ,

$$(19) \quad \sum_{k=0}^{\infty} |\mu_T|(R_{k+1} \setminus R_k) = |\mu_T|(\Gamma \setminus F) < +\infty,$$

and so, for every $\varepsilon > 0$, there exists an integer K such that

$$(20) \quad |\mu_T|((\Gamma \setminus F) \setminus R_K) = \sum_{k=K}^{\infty} |\mu_T|(R_{k+1} \setminus R_k) < \varepsilon.$$

By equation (18), the inequality in (20) implies that $\|\mu_T^\pm - \mu_k\| < \varepsilon$, for all $k > K$. Thus, $\mu_T^\pm \in A_\Gamma$, as desired, and there exists an element f in $L^1(S)$ such that $\mu_T^\pm = v_f$. Turning now to $\mu_T^0 = K(F)\mu_T$, recall that

$$F = \bigcup_{m=1}^N \Gamma'_m = \Gamma'_1 \cup (\Gamma'_2 \setminus \Gamma'_1) \cup \dots \cup \left(\Gamma'_m \setminus \left(\bigcup_{j=1}^{m-1} \Gamma'_j \right) \right) \cup \dots \cup \left(\Gamma'_N \setminus \left(\bigcup_{j=1}^{N-1} \Gamma'_j \right) \right).$$

Define measures τ_m , $1 \leq m \leq N$, in $M(\Gamma)$ as follows:

$$\tau_1 = K(\Gamma'_1)\mu_T, \quad \tau_2 = K(\Gamma'_2 \setminus \Gamma'_1)\mu_T, \quad \dots, \quad \tau_N = K\left(\Gamma'_N \setminus \left(\bigcup_{j=1}^{N-1} \Gamma'_j\right)\right)\mu_T.$$

Observe that $\mu_T^0 = \sum_{m=1}^N \tau_m$. Since $\mathfrak{M}(L^1(S))$ is a convolution measure algebra, each measure τ_m , $1 \leq m \leq N$, being absolutely continuous with respect to μ_T , corresponds to a multiplier of $L^1(S)$. It suffices therefore to determine the form of an arbitrary multiplier measure τ satisfying $\text{supp}(\tau) \subseteq \Gamma'_m$, for some m , $1 \leq m \leq N$.

Because $L^1(S) = L^1(a_m, b_m) \hat{\otimes} L^1(I'_m)$, without loss of generality, assume that $m=1$ and that the support of the multiplier measure τ is contained in Γ'_1 . It remains to prove that $\tau = \delta_1 \otimes \eta$, where δ_1 in $M(\Gamma_1)$ is the point mass at O_1 , and η is a measure in $M\left(\prod_{n=2}^N \Gamma_n\right)$ corresponding to a multiplier of $L^1(I'_1) = L^1\left(\prod_{n=2}^N I_n\right)$. Define η on the semicharacters $\psi_{x'} = \bigoplus_{n=2}^N \psi_{x_n}$, where $x' = (x_n)_{n=2}^N \in \prod_{n=2}^N (a_n, b_n]$, of $\prod_{n=2}^N \Gamma_n$ by the formula:

$$(21) \quad \langle \psi_{x'}, \eta \rangle = \langle \psi_{b_1} \otimes \psi_{x'}, \tau \rangle.$$

Since the linear span of such semicharacters $\psi_{x'}$ is dense in $C\left(\prod_{n=2}^N \Gamma_n\right)$, equation (21)

defines a unique measure η in $M\left(\prod_{n=2}^N \Gamma_n\right)$. For each $x=(x_n)_{n=1}^N$ in $\prod_{n=1}^N (a_n, b_n)$,

$$(22) \quad \begin{aligned} \langle \psi_x, \delta_1 \otimes \eta \rangle &= \langle \psi_{x_1}, \delta_1 \rangle \langle \psi_{x'}, \eta \rangle = \langle \psi_{x'}, \eta \rangle = \\ &= \langle \psi_{b_1} \otimes \psi_{x'}, \tau \rangle = \langle \psi_x, \tau \rangle + \langle (\psi_{b_1} - \psi_{x_1}) \otimes \psi_{x'}, \tau \rangle. \end{aligned}$$

A routine calculation shows that $\text{supp}((\psi_{b_1} - \psi_{x_1}) \otimes \psi_{x'}) \cap \text{supp}(\tau) = \emptyset$; hence, equation (22) reduces to

$$(23) \quad \langle \psi_x, \delta_1 \otimes \eta \rangle = \langle \psi_x, \tau \rangle, \quad \psi_x \in \hat{F},$$

implying that $\tau = \delta_1 \otimes \eta$. It remains to show that η corresponds to a multiplier of $L^1(I'_1)$. To accomplish this, fix f in $L^1(a_1, b_1)$ with $\hat{f}(b_1) = \int_{a_1}^{b_1} f(t_1) dt_1 = 1$. Then, for each g in $L^1(I'_1)$, $f \otimes g \in L^1(a_1, b_1) \otimes L^1(I'_1) \subseteq L^1(S)$, and so $v_{f \otimes g} \in A_\Gamma$. If we let $v' = \bigotimes_{n=2}^N v^{(n)}$ be the canonical imbedding of $L^1(I'_1) = \bigotimes_{n=2}^N L^1(a_n, b_n)$ in $\bigotimes_{n=2}^N A_{\Gamma_n}$, then, since $A_\Gamma = \bigotimes_{n=1}^N A_{\Gamma_n} = A_{\Gamma_1} \hat{\otimes} \left(\bigotimes_{n=2}^N A_{\Gamma_n} \right)$, it follows that $v = v^{(1)} \hat{\otimes} v'$. This decomposition of v reveals that $v_{f \otimes g} = v_f^{(1)} \otimes v'_g \in A_{\Gamma_1} \otimes \left(\bigotimes_{n=2}^N A_{\Gamma_n} \right)$, for every g in $L^1(I'_1)$. For each semi-character ψ_x , $x=(x_n) \in \prod_{n=1}^N (a_n, b_n]$, on Γ_1

$$(24) \quad \begin{aligned} \langle \psi_x, \tau * v_{f \otimes g} \rangle &= \langle \psi_{x_1} \otimes \psi_{x'}, \delta_1 \otimes \eta \times \psi_{x_1} \otimes \psi_{x'}, v_f^{(1)} \otimes v'_g \rangle = \\ &= \langle \psi_{x_1}, v_f^{(1)} \rangle \langle \psi_{x'}, \eta \rangle \langle \psi_{x'}, v'_g \rangle = \langle \psi_x, v_f^{(1)} \otimes (\eta * v'_g) \rangle; \end{aligned}$$

whence, it follows that

$$(25) \quad v_f^{(1)} \otimes (\eta * v'_g) = \tau * v_{f \otimes g},$$

for all g in $L^1(I'_1)$. Because τ is a multiplier measure on A_Γ and $v_{f \otimes g} \in A_\Gamma$, equation (25) implies that $v_f^{(1)} \otimes (\eta * v'_g) \in A_\Gamma$, for every g in $L^1(I'_1)$.

Now, $\Theta_1: A_{\Gamma_1} \hat{\otimes} \left(\bigotimes_{n=2}^N A_{\Gamma_n} \right) \rightarrow \bigotimes_{n=2}^N A_{\Gamma_n}$ defined by $\Theta_1 \left(\sum_{i=1}^\infty \gamma_i \otimes \alpha_i \right) = \sum_{i=1}^\infty \langle \psi_{h_1}, \gamma_i \rangle \alpha_i$ is a continuous linear map from $A_{\Gamma_1} \hat{\otimes} \left(\bigotimes_{n=2}^N A_{\Gamma_n} \right)$ to $\bigotimes_{n=2}^N A_{\Gamma_n}$ [9, Lemma 2]. Therefore, for each $g \in L^1(I'_1)$,

$$(26) \quad \Theta_1(v_f^{(1)} \otimes (\eta * v'_g)) = \eta * v'_g \in \bigotimes_{n=2}^N A_{\Gamma_n}.$$

Hence, η corresponds to a multiplier of $L^1(I'_1)$. ■

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