On a lemma of P. Erdős and A. Rényi about random graphs

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In 1959 P. ERDŐS and A. RÉNYI stated some theorems on random graphs using — as they called it — 'a rather surprising lemma' which is given below.

Lemma 1 *(Erdős—Rényi). Let Γ_{n,N_c} be a 'random graph' with n possible (labelled) vertices and N_c edges, i.e. all $\binom{\binom{n}{2}}{N_c}$ possible graphs are equiprobable. Further let $P(\bar{A}, n, N_c)$ denote the probability that Γ_{n,N_c} does not consist of one connected graph having $n-k$ effective vertices and k isolated points $(k=0, 1, 2, \ldots)$. Then for $N_c=\left[\frac{1}{2} n\log n+cn\right]$ it holds true that $\lim\limits_{n\to\infty} P(\bar{A}, n, N_c)=0$.*

PROOF See [2], 292—295.

This basic lemma has recently been used by WRIGHT (1976) to obtain further results about random graphs (see [3]). The first part of Erdős's and Rényi's proof was mainly based on the inequalities

$$a_n(s)=\binom{n}{s}\frac{\binom{\binom{n}{2}-s(n-s)}{N_c}}{\binom{\binom{n}{2}}{N_c}}\leqq\frac{e^{(3-2c)s}}{s!}\qquad\text{for}\quad s\leqq\frac{n}{2},\tag{1}$$

and

$$a_n(s)=\binom{n}{s}\frac{\binom{\binom{n}{2}-s(n-s)}{N_c}}{\binom{\binom{n}{2}}{N_c}}\leqq\frac{e^{(3-2c)(n-s)}}{(n-s)!}\qquad\text{for}\quad s\geqq\frac{n}{2},\tag{2}$$

which were pointed out to hold true for all $n\geqq n_0$, where n_0 may depend on c but not on s. Here (2) follows from (1) by symmetry.

But the following arguments indicate that the uniform estimates in (1) and (2) cannot be valid:

Let $t \in \left(0, \frac{1}{2}\right]$ be fixed, and consider the N_c-th root of

$$b_n(t) \equiv a_n([nt]) \Big/ \frac{e^{(3-2c)[nt]}}{[nt]!}.$$

Setting $a = \binom{n}{2}$, $d = [nt](n-[nt])$, $N_c = \left[\frac{1}{2} n \log n + cn\right]$, and using Stirling's formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, we get

$$b_n(t) = \frac{n!}{(n-[nt])!} \frac{(a-d)!}{(a-d-N_c)!} \frac{(a-N_c)!}{a!} e^{-(3-2c)[nt]} \sim$$

$$\sim \frac{n^{n+\frac{1}{2}} e^{-n} (a-d)^{a-d+\frac{1}{2}} e^{-a+d} (a-N_c)^{a-N_c+\frac{1}{2}} e^{-a+N_c} e^{-(3-2c)[nt]}}{(n-[nt])^{n-[nt]+\frac{1}{2}} e^{-n+[nt]} (a-d-N_c)^{a-d-N_c+\frac{1}{2}} e^{-a+d+N_c} a^{a+\frac{1}{2}} e^{-a}} =$$

$$= n^{[nt]} \left(\frac{1}{1-\frac{[nt]}{n}} \right)^{n-[nt]+\frac{1}{2}} \frac{\left(1-\frac{d}{a}\right)^{N_c} \left(1-\frac{N_c}{a}\right)^{a-N_c+\frac{1}{2}} e^{(2c-4)[nt]}}{\left(1-\frac{N_c}{a-d}\right)^{a-d-N_c+\frac{1}{2}}}.$$

From this

$$\lim_{n\to\infty} b_n(t) = e^{2t} \cdot 1 \cdot \frac{\left(1-2t(1-t)\right) e^{-1} \cdot 1}{e^{-1}} = \left(1-2t(1-t)\right) e^{2t} \equiv b(t)$$

is an immediate consequence. Since $b(0)=1$ and $b'(t)=4t^2 e^{2t} > 0$ $(t \neq 0)$, we have $b(t) > 1$ for $t \in \left(0, \frac{1}{2}\right]$, which contradicts (1), i.e. (1) cannot hold true for such s with $s=[nt]$ (t as above).

But though the inequalities (1) and (2) are false, Lemma 1 remains true since it suffices to find another sequence $\{a(s)\}_{s=0,1,2,\ldots}$ with $\sum_{s=0}^{\infty} a(s) < \infty$ such that (1) holds true if $e^{(3-2c)s}/s!$ is replaced by $a(s)$ (cf. [2], (16)). Such a sequence will be given in the following lemma.

Lemma 2. *There is an integer $n_0(c)$ such that for all $n \geqq n_0(c)$*

$$a_n(s) \leqq \frac{e^{(3-2c)s}}{s!} \quad \text{if} \quad s \leqq \frac{n}{\log n}, \tag{1a}$$

and

$$a_n(s) \leqq q^s \quad \text{if} \quad \frac{n}{\log n} \leqq s \leqq \frac{n}{2}, \tag{1b}$$

where $0 < q < 1$.

PROOF. Using an inequality in BOLLOBÁS and SAUER ([1], p. 1340) we find that

$$a_n(s) \leqq \frac{n^s}{s!} e^{-s(n-s)N_c/\binom{n}{2}}. \tag{3}$$

Underestimating $n-s$ by $n-\frac{n}{\log n}$ in the first case it follows immediately that (1a) holds.

For $\frac{n}{\log n} \leqq s \leqq \frac{n}{2}$ and n sufficient large the inequality

$$\frac{s(n-s)N_c}{\binom{n}{2}} \geqq \frac{1}{3} s \log n \tag{4}$$

follows. Furthermore Stirling's formula yields

$$s! \geqq s^s e^{-s}. \tag{5}$$

Taking (3), (4), and (5) together we have

$$a_n(s) \leqq \left(\frac{e \log n}{\sqrt[3]{n}}\right)^s$$

(for that range of s named above) from which (1b) is obvious.

Now, to retain Erdős's and Rényi's lemma, inequality (16) in [2] can be replaced by

$$P(\bar{E}_M, n, N_c) \leqq \sum_{s > \frac{2N_c}{n}} a(s) + \sum_{s > M} a(s) \quad \text{for} \quad n \geqq n_0,$$

where

$$a(s) = \frac{e^{(3-2c)s}}{s!} + q^s, \quad s = 0, 1, 2, \dots.$$

Then

$$\lim_{n\to\infty} P(\bar{E}_{\log\log n}, n, N_c) = 0$$

still holds, and the lemma remains true.

References

[1] B. Bollobás, N. Sauer, Uniquely colourable graphs with large girth. *Canad. J. Math.* **28** (1976), 1340—1344.

[2] P. Erdős, A. Rényi, On random graphs I. *Publ. Math. (Debrecen)* **6** (1959), 290—297.

[3] E. M. Wright, The evolution of unlabelled graphs. *J. London Math. Soc., Ser. 2*, **14** (1976), 554—558.

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