Differential-geometric Properties of Indicatrix Bundle over Finsler Space

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To study indicatrices of a Finsler space is essential for studying the space itself, as it has been emphasized by many authors (cf. [6]). It should be, however, awaken that by studying indicatrices we can only see the behavior of tensors derived from the fundamental function by differentiating with respect to the supporting element alone. Therefore we naturally intend to study the set of indicatrices at all points of the space. The set is regarded as a hypersurface of the tangent bundle, and so we need the differential geometry of tangent bundle.

The author has studied the tangent bundle over a Finsler space ([2], [3], [4]). In the present paper the tangent bundle is regarded as a differentiable manifold which is equipped with a Riemannian metric called the 0-lift and a linear connection derived from the Cartan connection. Although certain noteworthy results are obtained in the present paper, it seems to the author that various new questions have arisen one after another by the present studies.

All preliminary concepts are systematically described in the monograph [7]. The quotation from the monograph is indicated by putting asterisk.

§ 1. Preliminaries

This section consists of extracts from *Chapter IV (concerned with the differential geometry of tangent bundle) of the monograph [7] and certain additional remarks.

(1) N-homomorphism

By $F(F^n)$ is denoted the Finsler bundle of an n-dimensional Finsler space F^n, i.e., the induced bundle $\pi_T^{-1}(L(F^n))$ from the linear frame bundle $L(F^n)$ over F^n by the projection π_T: $T \to F^n$ of the tangent bundle $T(F^n)$ over F^n. Next $\bar{L}(T)$ indicates the linear frame bundle over T. If a non-linear connection N is given in T, we get a bundle homomorphism φ_N: $F \to \bar{L}$, defined by $\varphi_N(u) = (l_y(z), l_y^v(z))$ for $u = (y, z) \in F$, where l_y is the (horizontal) lift with respect to the N and l_y^v is the vertical lift. With this bundle homomorphism φ_N the group homomorphism ψ: $G(=GL(n, R)) \to \bar{G}(=GL(2n, R))$ is associated, where $\psi(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$ for $g \in G$. The φ_N is called the *N-homomorphism*.

From a local coordinate (x^i) of $x \in F^n$ we obtain the induced coordinates[1] $(\bar{x}^\lambda) = (\bar{x}^i, \bar{x}^{(i)}) = (x^i, y^i)$ of $y \in T$ and (x^i, y^i, z^i_a) of $u = (y, z) \in F$, where $y = y^i(\partial/\partial x^i)_x$, $z = (z_a)$ and $z_a = z^i_a(\partial/\partial x^i)$. From the induced coordinate $(\bar{x}^\lambda)$ the coordinate $(\bar{x}^\lambda, \bar{z}^\lambda_\alpha)$ of $\bar{z} \in \bar{L}$ is induced. In terms of these coordinates the N-homomorphism φ_N is written as follows: For a point $u = (y, z) = (x^i, y^i, z^i_a) \in F$ the coordinate of $\varphi_N(u) = \bar{z}$ is $(x^i, y^i, \bar{z}^\lambda_\alpha)$, where $\bar{z}^\lambda_\alpha$ are given by

$$(1.1) \qquad (\bar{z}^i_a, \bar{z}^{(i)}_{(a)}, \bar{z}^i_{(a)}, \bar{z}^{(i)}_{(a)}) = (z^i_a, -z^j_a N^i{}_j, 0, z^i_a).$$

Real valued functions $N^i{}_j$ on the domain of coordinate (x^i, y^i) are connection parameters of the non-linear connection N. Let $(\bar{z}^{-1})^\alpha_\lambda$ be elements of the inverse matrix of $(\bar{z}^\lambda_\alpha)$. Then we get

$$(1.2) \qquad \left((\bar{z}^{-1})^a_i, (\bar{z}^{-1})^a_{(i)}, (\bar{z}^{-1})^{(a)}_i, (\bar{z}^{-1})^{(a)}_{(i)}\right) = \left((z^{-1})^a_i, 0, (z^{-1})^a_j N^j{}_i, (z^{-1})^a_i\right),$$

where $(z^{-1})^a_i$ are elements of the inverse matrix of (z^i_a).

(2) N-decompositions

By an N-homomorphism φ_N Finsler tensor fields are derived from a tensor field on T as follows: Let V^r_s be the tensor space of (r, s)-type, constructed from the real vector n-space V and its dual space V^* by tensor product. Then a Finsler tensor field K of (r, s)-type is regarded as a V^r_s-valued function on F, satisfying the equation $K \cdot \beta_g = g^{-1} K$ for any $g \in G$, where β_g is the right-translation of F by g. Similarly a tensor field $\bar{K}$ of (r, s)-type on T is regarded as a $\bar{V}^r_s$-valued function on $\bar{L}$, satisfying $\bar{K} \cdot \gamma_{\bar{g}} = \bar{g}^{-1} \bar{K}$ for any $\bar{g} \in \bar{G}$, where $\bar{V} = V \times V$ and $\gamma_{\bar{g}}$ is the right-translation of $\bar{L}$ by $\bar{g}$. Now, from $\bar{K}$ of $(1, 1)$-type, for instance, four V^1_1-valued functions K^ξ_η, $\xi, \eta = 1, 2$, on F are introduced by

$$(1.3) \qquad \begin{aligned} K^1_1(v^*, v) &= (\bar{K} \cdot \varphi_N)\big((v^*, 0),\ (v, 0)\big), \\ K^1_2(v^*, v) &= (\bar{K} \cdot \varphi_N)\big((v^*, 0),\ (0, v)\big), \\ K^2_1(v^*, v) &= (\bar{K} \cdot \varphi_N)\big((0, v^*),\ (v, 0)\big), \\ K^2_2(v^*, v) &= (\bar{K} \cdot \varphi_N)\big((0, v^*),\ (0, v)\big), \end{aligned}$$

for any $v \in V$ and any $v^* \in V^*$, where $(v, 0), (0, v) \in V \times V$ and $(v^*, 0), (0, v^*) \in V^* \times V^*$. These K^ξ_η are Finsler tensor fields of $(1, 1)$-type and called *N-decompositions* of $\bar{K}$.

As to a tangent vector field X of T, its N-decompositions X^1 and X^2 are respectively horizontal and vertical components of X with respect to the N. That is, putting $X = X^i(\partial/\partial x^i) + X^{(i)}(\partial/\partial y^i)$, we have

$$X = X^i(\partial/\partial x^i - N^j{}_i \partial/\partial y^j) + (X^{(i)} + N^i{}_j X^j)(\partial/\partial y^i),$$

where X^i (resp. $X^{(i)} + N^i{}_j X^j$) are components of X^1 (resp. X^2).

[1]) Latin indices run from 1 to n, Greek indices run from 1 to $2n$ and $(i) = n + i$ throughout.

(3) Linear connections of Finsler type

Suppose that a Finsler connection $F\Gamma=(\Gamma, N)$ be given on the F^n; Γ is a connection in $F(F^n)$ and N is a non-linear connection in $T(F^n)$. By this N we get an N-homomorphism φ_N and N-decompositions as above mentioned. Further a connection $\varphi_N(\Gamma)$ is induced in $\bar{L}$. This is a linear connection on T, said to be *of Finsler type* and denoted by $F\bar{\Gamma}$. In the following we are mainly concerned with the Cartan connection $C\Gamma$; $\varphi_N(\Gamma)$ is called *of Cartan type* and denoted by $C\bar{\Gamma}$.

The relation between h-basic vector field $B^h(v)$ (resp. v-basic vector field $B^v(v)$) of $F\Gamma$ and basic vector field $\bar{B}(\bar{v})$ of $F\bar{\Gamma}$ is given by $\varphi'_N(B^h(v))=\bar{B}((v,0))$ (resp. $\varphi'_N(B^v(v))=\bar{B}((0,v))$), which implies explicit form of connection parameters $\bar{\Gamma}_\mu{}^\lambda{}_\nu$ of $F\bar{\Gamma}$ in a coordinate (x^i, y^i) as follows:

$$(1.4)\qquad \begin{aligned} &\bar{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k, \quad \bar{\Gamma}_{jk}^{(i)} = \partial_k N^i{}_j + N^r{}_j \Gamma_r{}^i{}_k - N^i{}_r \Gamma_j{}^r{}_k,\\ &\bar{\Gamma}_{(j)k}^{\ i} = 0, \qquad \bar{\Gamma}_{(j)k}^{(i)} = \Gamma_j{}^i{}_k\\ &\bar{\Gamma}_{j(k)}^{\ i} = C_j{}^i{}_k, \quad \bar{\Gamma}_{j(k)}^{(i)} = \dot{\partial}_k N^i{}_j + N^r{}_j C_r{}^i{}_k - N^i{}_r C_j{}^r{}_k,\\ &\bar{\Gamma}_{(j)}{}^i{}_{(k)} = 0, \quad \bar{\Gamma}_{(j)}{}^{(i)}{}_{(k)} = C_j{}^i{}_k, \end{aligned}$$

where $\Gamma_j{}^i{}_k = F_j{}^i{}_k + C_j{}^i{}_r N^r{}_k$ (cf. *(9.3)).

N-decompositions of the torsion tensor $\bar{T}$ and the curvature tensor $\bar{R}$ of $C\bar{\Gamma}$ are given by

$$(1.5)\qquad (\bar{T}_1{}^1{}_1, \bar{T}_1{}^2{}_1, \bar{T}_1{}^1{}_2, \bar{T}_1{}^2{}_2, \bar{T}_2{}^1{}_2, \bar{T}_2{}^2{}_2) = (0, R^1, C, P^1, 0, 0),$$

$$(1.6)\qquad (\bar{R}_\eta{}^\xi{}_{11}, \bar{R}_\eta{}^\xi{}_{12}, \bar{R}_\eta{}^\xi{}_{22}) = (\delta^\xi_\eta R^2, \delta^\xi_\eta P^2, \delta^\xi_\eta S^2), \quad \xi, \eta = 1, 2.$$

Remark. $\bar{T}_1{}^2{}_1 = R^1$, for instance, means $\bar{T}_b{}^{(a)}{}_c = R^a{}_{bc} \in V^1_2$, but not $\bar{T}_j{}^{(i)}{}_k = R^i{}_{jk}$ in our usage of indices. In fact, from (1.4), we get

$$\begin{aligned} \bar{T}_j{}^{(i)}{}_k &= \partial_k N^i{}_j - \partial_j N^i{}_k + N^r{}_j \Gamma_r{}^i{}_k - N^r{}_k \Gamma_r{}^i{}_j - N^i{}_r (\Gamma_j{}^r{}_k - \Gamma_k{}^r{}_j) = \\ &= R^i{}_{jk} + P^i{}_{jr} N^r{}_k - P^i{}_{kr} N^r{}_j + N^i{}_r (C_k{}^r{}_s N^s{}_j - C_j{}^r{}_s N^s{}_k). \end{aligned}$$

§ 2. The normal vector and tangent space of indicatrix bundle

We are concerned with an n-dimensional Finsler space F^n with a fundamental function $L(x, y)$ [1]) and the Cartan connection $C\Gamma=(\Gamma, N)$. For a fixed point $x=(x^i)\in F^n$ the equation $L(x, y)=1$ defines a hypersurface I_x of the tangent space F^n_x of F^n at x, which is called the *indicatrix* at x.

Consider the tangent bundle $T(F^n)$ over F^n. In the total space T the equation $L(x, y)=1$ defines a hypersurface, consisting of indicatrices at all points of F^n.

Definition. A hypersurface $I(F^n)$ of the tangent bundle $T(F^n)$ over a Finsler space F^n given by the equation $L(x, y)=1$ is called the *indicatrix bundle* over F^n. The intersection $I(F^n_x)$ of $I(F^n)$ with the fibre over a point $x\in F^n$ is called the *indicatrix fibre* over x.

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[1]) We abbreviate $L(x^1, \ldots, x^n, y^1, \ldots, y^n)$ to $L(x, y)$.

That is, $I(F^n_x)$ is a subspace of T given by the equation $L(x, y)=1$ for a fixed point x and is essentially equal to the indicatrix I_x.

We take $2n-1$ variables u^I ($I, J, K, \ldots$ run from 1 to $2n-1$ throughout), in terms of which the indicatrix bundle $I(F^n)$ is given by parametric equations

$$x^i = x^i(u^I), \quad y^i = y^i(u^I), \tag{2.1}$$

where the matrix

$$(B^\lambda_{I)}) = (B^i_{I)}, B^{(i)}_{I)}) = (\partial x^i/\partial u^I, \partial y^i/\partial u^I) \tag{2.2}$$

is of rank $2n-1$. These $2n-1$ vectors $B_{I)}$ with components $B^\lambda_{I)}$ are linearly independent and span the tangent space of $I(F^n)$. Denoting N-decompositions $(B^1_{I)}, B^2_{I)})$ of $B_{I)}$ by $(C_{I)}, D_{I)})$ for brevity, we have

$$(C^i_{I)}, D^i_{I)}) = (B^i_{I)}, B^{(i)}_{I)} + N^i{}_r B^r_{I)}). \tag{2.2'}$$

Throughout the paper we are concerned with the space $(T, \bar{g}, C\bar{\Gamma})$, i.e., the tangent space T equipped with the 0-lift $\bar{g}$ and the linear connection of Cartan type $C\bar{\Gamma}$. The 0-lift $\bar{g}$ is by definition a Riemannian metric of T, whose N-decompositions are

$$(\bar{g}_{11}, \bar{g}_{12}, \bar{g}_{22}) = (g, 0, g), \tag{2.3'}$$

where g is the fundamental tensor of F^n (*§ 21). Components of $\bar{g}$ in a coordinate (x^i, y^i) are given by

$$(\bar{g}_{ij}, \bar{g}_{i(j)}, \bar{g}_{(i)(j)}) = (g_{ij} + g_{rs} N^r{}_i N^s{}_j, N^r{}_i g_{rj}, g_{ij}), \tag{2.3}$$

where $N^i{}_j$ are, of course, connection parameters of the non-linear connection N of $C\Gamma=(\Gamma, N)$. Contravariant components of $\bar{g}$ are

$$(\bar{g}^{ij}, \bar{g}^{i(j)}, \bar{g}^{(i)(j)}) = (g^{ij}, -g^{ir} N^j{}_r, g^{ij} + g^{rs} N^i{}_r N^j{}_s), \tag{2.4}$$

and N-decompositions are

$$(\bar{g}^{11}, \bar{g}^{12}, \bar{g}^{22}) = (g^{-1}, 0, g^{-1}), \tag{2.4'}$$

where g^{-1} indicates the contravariant fundamental tensor. The scalar product $\bar{g}(X, Y)$ of two tangent vectors X, Y of T is given by $g(X^1, Y^1)+g(X^2, Y^2)$. It then follows that, with respect to the 0-lift $\bar{g}$, the non-linear connection N_y is orthogonal to the vertical subspace T^v_y, and if tangent vectors X, Y of F^n are orthogonal with respect to a supporting element y, then $l_y(X)$ (resp. $l^v_y(X)$) is orthogonal to $l_y(Y)$ (resp. $l^v_y(Y)$).

It should be emphasized that $C\bar{\Gamma}$ is metrical with respect to $\bar{g}$ (*Theorem 21.1), although it is not the Riemannian connection determined by $\bar{g}$, provided that F^n is not Riemannian, as it is known from (1.5).

From the equation $L(x, y)=1$ of $I(F^n)$ it follows that $(B_\lambda)=(\partial L/\partial x^i, \partial L/\partial y^i)$ are components of a covariant normal vector B_* of $I(F^n)$. From $L_{|i}=0$ and $\partial L/\partial y^i = l_i$ we have components of B_*:

$$B_* = (l_r N^r{}_i, l_i), \tag{2.5}$$

and N-decompositions (C_*, D_*) of B_* are

$$(C_{*i}, D_{*i}) = (0, l_i). \tag{2.5'}$$

From (2.2′) and (2.5′) the equation $\partial L/\partial u^I = 0$ is written as

$$l_i D^i_{I)} = 0. \tag{2.6}$$

From (2.4′) and (2.5′) it is seen that N-decompositions (C, D) of the contravariant normal vector B, corresponding to B_* by the 0-lift $\bar{g}$, are given by

$$(C^i, D^i) = (0, l^i) \tag{2.7′}$$

and components are

$$B = (0, l^i). \tag{2.7}$$

We recall the *intrinsic vertical vector field* l^v (*Definition 3.5), which is a vertical vector field on T having the value $(l^v)_y = l^v_y(y)$ at a point y. It is written as $y^i(\partial/\partial y^i)_y$. Thus (2.7) or (2.7′) shows

Proposition 1. *In the tangent bundle $(T, \bar{g}, C\bar{\Gamma})$ over F^n the intrinsic vertical vector field l^v is a unit normal vector field of the indicatrix bundle.*

Remark. The above shows a conspicuous property of the 0-lift $\bar{g}$. There will be many varieties to introduce a Riemannian metric in T from the fundamental tensor g of F^n (cf. *Definition 21.1 and [1]) and we applied $\bar{g}$ in the present paper for simplicity alone. It may be said that Proposition 1 is just a reason why the 0-lift is applied (cf. *Theorems 21.1 and 23.2).

It is well-known (cf. *§ 31 and [6]) that l^i is a unit normal vector of the indicatrix. Therefore, on account of properties of $\bar{g}$, we are led to

Theorem 1. *Let $(I_x)_y$ be the tangent space at a point y of the indicatrix I_x. Then the tangent space $(I(F^n))_y$ of the indicatrix bundle $I(F^n)$ at y is written in a direct sum $N_y \oplus V_y$, where N_y is the value of the non-linear connection N of the Cartan connection $C\Gamma = (\Gamma, N)$ and V_y is the vertical lift of $(I_x)_y$. The tangent space of the indicatrix fibre $I(F^n_x)$ is V_y.*

We have the inverse matrix $(B^{I)}, B_*)$ of $(B_{I)}, B)$. The $2n-1$ covariant vectors $B^{I)} = (B^{I)}_\lambda)$ are determined by equations

$$(1)\quad B^{I)}_\lambda B^\lambda_{J)} = \delta^I_J, \qquad (2)\quad B^{I)}_\lambda B^\lambda = 0. \tag{2.8}$$

Paying attention to (2.7′), in terms of N-decompositions $(C^{I)}, D^{I)})$ of $B^{I)}$, (2.8) is written in the form

$$(1)\quad C^{I)}_i C^i_{J)} + D^{I)}_i D^i_{J)} = \delta^I_j, \qquad (2)\quad D^{I)}_i l^i = 0. \tag{2.8′}$$

The inverse property of $(B^{I)}, B_*)$ is also written in the form $B^\lambda_{I)} B^{I)}_\mu + B^\lambda B_\mu = \delta^\lambda_\mu$. Therefore, introducing a tensor $\bar{h}$ by

$$\bar{h}^\lambda{}_\mu = \delta^\lambda_\mu - B^\lambda B_\mu, \tag{2.9}$$

we have

$$B^\lambda_{I)} B^{I)}_\mu = \bar{h}^\lambda{}_\mu. \tag{2.10}$$

From (2.5′), (2.7′) and the fact that N-decompositions of the Kronecker delta δ^λ_μ are $(\delta^1_1, \delta^1_2, \delta^2_1, \delta^2_2)=(\delta^i_j, 0, 0, \delta^i_j)$, we have N-decompositions of $\bar{h}$ as follows:

$$((\bar{h}^1{}_1)^i{}_j, (\bar{h}^1{}_2)^i{}_j, (\bar{h}^2{}_1)^i{}_j, (\bar{h}^2{}_2)^i{}_j) = (\delta^i_j, 0, 0, h^i{}_j), \tag{2.9′}$$

where $h^i{}_j=\delta^i_j-l^i l_j$ and $h_{ij}=g_{ik}h^k{}_j=(\dot{\partial}_i\dot{\partial}_j L)/L$ are components of the so-called *angular metric tensor*. Thus $\bar{h}$ is called the *lifted angular metric tensor*.

From (2.9′) we have the N-decomposition form of (2.10):

$$C^i_{I)}C^{I)}_j = \delta^i_j, \qquad D^i_{I)}D^{I)}_j = h^i{}_j, \qquad C^i_{I)}D^{I)}_j = D^i_{I)}C^{I)}_j = 0. \tag{2.10′}$$

From the Riemannian metric $\bar{g}$ of T is induced on $I(F^n)$ a Riemannian metric $\underline{g}$ with components

$$\underline{g}_{IJ} = \bar{g}_{\lambda\mu}B^\lambda_{I)}B^\mu_{J)} = g(C_{I)}, C_{J)})+g(D_{I)}, D_{J)}). \tag{2.11}$$

From (2.10′) and (2.11) we have

$$\underline{g}_{IJ}C^{I)}_i C^{J)}_j = g_{ij}, \qquad \underline{g}_{IJ}C^{I)}_i D^{J)}_j = 0, \qquad \underline{g}_{IJ}D^{I)}_i D^{J)}_j = h_{ij}. \tag{2.12}$$

Introducing $2n$ tangent vectors of $I(F^n)$ by

$$N_{i)} = C^{I)}_i B_{I)}, \qquad V_{i)} = D^{I)}_i B_{I)}, \tag{2.13}$$

the equations (2.12) shows

$$\underline{g}(N_{i)}, N_{j)}) = g_{ij}, \quad \underline{g}(N_{i)}, V_{j)}) = 0, \quad \underline{g}(V_{i)}, V_{j)}) = h_{ij}. \tag{2.14}$$

Now (2.10′) shows that N-decompositions of $N_{i)}$ and $V_{i)}$ are given by

$$((N^1_{i)})^j, (N^2_{i)})^j) = (\delta^j_i, 0), \quad ((V^1_{i)})^j, (V^2_{i)})^j) = (0, h^i{}_j). \tag{2.13′}$$

Consequently $N_{i)}$ and $V_{i)}$ are written in the coordinate as

$$N_{i)} = \partial/\partial x^i - N^j{}_i\partial/\partial y^j, \quad V_{i)} = h^j{}_i\partial/\partial y^j,$$

which imply that n independent vectors $N_{i)}$ span the subspace N_y of the tangent space of $I(F^n)$, and there are $n-1$ independent vectors among $V_{i)}$ (cf. *Proposition 16.2) which span the subspace V_y.

§ 3. Indicatory property in the tangent bundle

In *§ 31 we have introduced the concept of *indicatory property* of tensor on F^n. A similar property is now introduced for tensor on $T(F^n)$. We have the intrinsic vertical vector field l^v of $T(F^n)$ whose components as well as N-decompositions are $(0, y^i)$ at a point $y=(x^i, y^i)$. With respect to the 0-lift $\bar{g}$ we get the covariant vector field l^v_*, corresponding to l^v, whose components are $(y_r N^r{}_i, y_i)$ $(y_i=g_{ij}y^j)$ and N-decompositions are $(0, y_i)$.

Definition. A tensor $T^\lambda{}_\mu$ of (1, 1)-type, for instance, on $T(F^n)$ is called *indicatory* in the index μ (resp. λ), if $T^\lambda{}_\mu(l^v)^\mu=0$ (resp. $T^\lambda{}_\mu(l^v_*)_\lambda=0$). If a tensor on $T(F^n)$ is indicatory in every index, then it is called indicatory.

From the form of N-decompositions of l^v and l^v_* the indicatory property is expressed in terms of N-decompositions as follows:

Proposition 2. *A tensor $T^\lambda{}_\mu$ of (1, 1)-type, for instance, on $T(F^n)$ is indicatory in the index μ (resp. λ), if and only if its N-decompositions $(T^1{}_2)^i{}_j$ and $(T^2{}_2)^i{}_j$ (resp. $(T^2{}_1)^i{}_j$ and $(T^2{}_2)^i{}_j$) are indicatory in the index j (resp. i) in the sense of F^n.*

From (1.5), Proposition 2 and well-known properties of torsion tensors of $C\Gamma$ we have

Theorem 2. *(1) The torsion tensor $\bar{T}$ of the linear connection of Cartan type $C\bar{\Gamma}$ is indicatory. (2) The curvature tensor $\bar{R}_\varkappa{}^\lambda{}_{\mu\nu}$ of $C\bar{\Gamma}$ is indicatory in the indices μ and ν. It is indicatory in $\varkappa$ and λ, if and only if the $(v)h$- and $(v)hv$-torsion tensors R^1, P^1 of $C\Gamma$ vanish.*

Remark. It is easily verified that the condition $R^1=P^1=0$ is equivalent to $R^2=P^2=0$. Thus a problem "*Consider a class of Finsler spaces with $R_{hijk}=P_{hijk}=0$*", mentioned in the last remark of *§ 30, arises again in the viewpoint of the present paper.

The lifted angular metric tensor $\bar{h}$ plays an important role in the procedure of indicatorization, similarly to the angular metric tensor h in F^n (cf. *Definition 31.3). In fact, from the indicatory property of h and (2.9′) it follows that $\bar{h}$ is indicatory in $T(F^n)$; hence the procedure

$$T^\lambda{}_\mu \to {}'T^\lambda{}_\mu = T^\nu{}_\varkappa \bar{h}^\lambda{}_\nu \bar{h}^\varkappa{}_\mu \tag{3.1}$$

yields an indicatory tensor $'T$, called the *indicatorized tensor* of T. The following is easily verified by (2.9)′:

Proposition 3. *N-decompositions of the indicatorized tensor $'T$ of a tensor T of (1, 1)-type, for instance, are given by*

$$('T^1{}_1)^i{}_j = (T^1{}_1)^i{}_j, \qquad ('T^1{}_2)^i{}_j = (T^1{}_2)^i{}_r h^r{}_j,$$
$$('T^2{}_1)^i{}_j = (T^2{}_1)^r{}_j h^i{}_r, \quad ('T^2{}_2)^i{}_j = (T^2{}_2)^r{}_s h^i{}_r h^s{}_j.$$

It is easy to show that the indicatorized tensor $'\bar{g}$ is nothing but $\bar{h}$. Further $'l^v$ vanishes and $B_{I)}$ is indicatory; these are rather obvious results, because an indicatory tensor on $T(F^n)$ is regarded as the one on $I(F^n)$. Corresponding to *Proposition 31.2, we have

Proposition 4. *Let $T^\lambda{}_\mu$ be a tensor of (1, 1)-type, for instance, on $T(F^n)$, and let $T^I{}_J$ be the projection of $T^\lambda{}_\mu$ into $I(F^n)$, i.e., $T^I{}_J = T^\lambda{}_\mu B^{I)}_\lambda B^\mu_{J)}$. Then, as for the indicatorization $'T^\lambda{}_\mu$, we have $T^I{}_J = {}'T^\lambda{}_\mu B^{I)}_\lambda B^\mu_{J)}$ and $'T^\lambda{}_\mu = T^I{}_J B^\lambda_{I)} B^{J)}_\mu$.*

By introducing the Riemannian connection Γ^r from the 0-lift $\bar{g}$, the tangent bundle T is regarded as a Riemannian space $(T, \bar{g}, \Gamma^r)$. Thus the strain tensor S of $C\Gamma$ is defined by the equation $B^r(\bar{v}) = \bar{B}(\bar{v}) + \bar{Z}(S(\bar{v}))$ (*Definition 21.2), where $B^r(\bar{v})$ is the basic vector field of Γ^r, corresponding to $\bar{v} \in \bar{V}$, and $\bar{Z}(\bar{A})$ is the fundamental vector field on T, corresponding to an element $\bar{A}$ of the Lie algebra of $\bar{G}$. Because the covariant strain tensor S_* is written as *(21.9) in terms of the covariant torsion tensor $\bar{T}_*$, from Theorem 2 we have

Theorem 3. *The strain tensor S of the Cartan connection $C\Gamma$ is indicatory.*

§ 4. The second fundamental tensor of indicatrix bundle

The indicatrix bundle $I(F^n)$ is regarded as a hypersurface of the space $(T, \bar{g}, C\bar{\Gamma})$ and the vector field $B(=l^v)$ is a unit normal vector of $I(F^n)$. Therefore a connection $\underline{\Gamma}$ is induced in $I(F^n)$ from $C\bar{\Gamma}$ by the so-called Gauss formula

$$\partial B^\lambda_{I)}/\partial u^J + \bar{\Gamma}_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} = \underline{\Gamma}_I{}^K{}_J B^\lambda_{K)} + H_{IJ} B^\lambda, \tag{4.1}$$

where $\underline{\Gamma}_I{}^K{}_J$ are connection parameters of $\underline{\Gamma}$ and H_{IJ} are components of the second fundamental tensor H of $I(F^n)$ in $(T, \bar{g}, C\bar{\Gamma})$. From (4.1) we have

$$\bar{T}_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} = \underline{T}_I{}^K{}_J B^\lambda_{K)} + (H_{IJ} - H_{JI}) B^\lambda, \tag{4.2}$$

where $\underline{T}_I{}^K{}_J$ are components of the torsion tensor $\underline{T}$ of $\underline{\Gamma}$.

It should be remarked that the well-known theory of subspaces of a Riemannian space can not be carelessly applied to $I(F^n)$, because $C\bar{\Gamma}$ has the surviving torsion tensor $\bar{T}$ in general. It then seems from (4.2) that H is not a symmetric tensor, but it follows from Theorem 2 that (4.2) leads to the symmetry $H_{IJ} = H_{JI}$ by contracting by B_λ.

Differentiating (2.11) by u^K, it is seen that the induced connection $\underline{\Gamma}$ is metrical with respect to the induced metric $\underline{g}$. Further, differentiating equations $\bar{g}_{\lambda\mu} B^\lambda B^\mu_{I)} = 0$ and $\bar{g}_{\lambda\mu} B^\lambda B^\mu = 1$ by u^J, and substituting from (4.1), we obtain the so-called Weingarten formula

$$\partial B^\lambda/\partial u^J + \bar{\Gamma}_\mu{}^\lambda{}_\nu B^\mu B^\nu_{J)} = -H^K{}_J B^\lambda_{K)}, \tag{4.3}$$

where $H_{IJ} = \underline{g}_{IK} H^K{}_J$.

We shall write (4.1) and (4.3) in terms of N-decompositions of $B_{I)}$ and B. To do so, we put

$$E_j{}^i{}_K = F_j{}^i{}_k C^k_{K)} + C_j{}^i{}_k D^k_{K)} \left(= \Gamma_j{}^i{}_k B^k_{K)} + C_j{}^i{}_k B^{(k)}_{K)}\right). \tag{4.4}$$

Then, from (1.4) and (2.2′) it is seen that (4.1) is written as

$$\begin{aligned} &(1)\quad \partial C^i_{I)}/\partial u^J + E_j{}^i{}_J C^j_{I)} = \underline{\Gamma}_I{}^K{}_J C^i_{K)}, \\ &(2)\quad \partial D^i_{I)}/\partial u^J + E_j{}^i{}_J D^j_{I)} = \underline{\Gamma}_I{}^K{}_J D^i_{K)} + H_{IJ} D^i. \end{aligned} \tag{4.1′}$$

Similarly (4.3) is written in the form

$$(1)\quad H^K{}_J C^i_{K)} = 0, \qquad (2)\quad \partial D^i/\partial u^J + E_j{}^i{}_J D^j = -H^K{}_J D^i_{K)}. \tag{4.3′}$$

In both equations (4.1′) and (4.3′) the first equations are simpler than the second on account of $C^i = 0$.

From $D^i = l^i$ the second of (4.3′) is rewritten as

$$\begin{aligned} -H^K{}_J D^i_{K)} &= \partial l^i/\partial u^J + (F_j{}^i{}_k C^k_{J)} + C_j{}^i{}_k D^k_{J)}) l^j = \\ &= [(h^i{}_r N^r{}_j - N^i{}_j) B^j_{J)} + h^i{}_j B^{(j)}_{J)}] + N^i{}_k C^k_{J)}. \end{aligned}$$

From (2.2′) the above is written as $H^K{}_J D^i_{K)} = -h^i{}_j D^j_{J)}$. This, (1) of (4.3′) and (2.8′) yield

$$H^K{}_J \delta^I_K = H^K{}_J (C^{I)}_i C^i_{K)} + D^{I)}_i D^i_{K)}) = -D^{I)}_i h^i{}_j D^j_{J)},$$

which immediately implies

$$H_{IJ} = -h_{ij} D^i_{I)} D^j_{J)}, \tag{4.5}$$

or, from $h_{ij} = g_{ij} - l_i l_j$ and (2.6) we have

Proposition 5. *The second fundamental tensor H_{IJ} of the indicatrix bundle $I(F^n)$ as a hypersurface of the space $(T, \bar{g}, C\bar{\Gamma})$ is given by $H_{IJ} = -g(D_{I)}, D_{J)})$.*

Remark. We shall find the second fundamental tensor of $I(F^n)$ as a hypersurface of the Riemannian space $(T, \bar{g}, \Gamma^r)$. The difference between the connections $C\bar{\Gamma}$ and Γ^r is given by the strain tensor S: $\bar{\gamma}_\mu{}^\lambda{}_\nu = \bar{\Gamma}_\mu{}^\lambda{}_\nu - S_\mu{}^\lambda{}_\nu$, where $\bar{\gamma}_\mu{}^\lambda{}_\nu$ are connection parameters of Γ^r and $S_\mu{}^\lambda{}_\nu$ are components of the strain tensor S. Then (4.1) is rewritten as

$$\partial B^\lambda_{I)}/\partial u^J + \bar{\gamma}_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} = \underline{\Gamma}_I{}^K{}_J B^\lambda_{K)} + H_{IJ} B^\lambda - S_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)}.$$

Putting

$$\underline{S}_I{}^K{}_J = S_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} B^{K)}_\lambda, \quad \underline{\gamma}_I{}^K{}_J = \underline{\Gamma}_I{}^K{}_J - \underline{S}_I{}^K{}_J,$$

we obtain $S_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} = \underline{S}_I{}^K{}_J B^\lambda_{K)}$ from Theorem 3 and Proposition 4, and (4.1) is written in the form

$$\partial B^\lambda_{I)}/\partial u^J + \bar{\gamma}_\mu{}^\lambda{}_\nu B^\mu_{I)} B^\nu_{J)} = \underline{\gamma}_I{}^K{}_J B^\lambda_{K)} + H_{IJ} B^\lambda. \tag{4.1″}$$

Consequently *the second fundamental tensor of $I(F^n)$ as a hypersurface of the Riemannian space $(T, \bar{g}, \Gamma^r)$ is also given by $H_{IJ} = -g(D_{I)}, D_{J)})$.*

Now we consider the second fundamental tensor H in detail. From (2.11) and Proposition 5 we have

$$H_{IJ} + g_{IJ} = g(C_{I)}, C_{J)}). \tag{4.6}$$

From this, (4.5) and (2.10′) interesting equations are obtained:

$$(1)\quad H_{IJ} C^{J)}_i = 0, \qquad (2)\quad (H_{IJ} + g_{IJ}) D^{J)}_i = 0. \tag{4.7}$$

By (2.13) we already introduced $2n$ tangent vectors $N_{i)}$ and $V_{i)}$ which have respective components $C^{I)}_i$ and $D^{I)}_i$ in the coordinate (u^I) of $I(F^n)$. Then (4.7) shows the geometrical meaning that $N_{i)}$ are principal directions, corresponding to the principal curvature 0, and $V_{i)}$ are those, corresponding to the principal curvature -1. Consequently we have

Theorem 4. *At every point y of the indicatrix bundle $I(F^n)$ we have n-ple principal curvature 0 and $(n-1)$-ple one -1. The principal directions, corresponding to the principal curvature 0 (resp. -1), constitute the subspace N_y (resp. V_y) in Theorem 1.*

Remark. Theorem 4 shows that $I(F^n)$ is something like a cylinder defined by the equation $(y^1)^2 + \ldots + (y^n)^2 = 1$ in the $2n$-dimensional euclidean space with a orthonormal coordinate (x^i, y^i) in the view point of principal curvature.

§ 5. The torsion and curvature of indicatrix bundle

First we consider the torsion tensor $\underline{T}$. The equation (4.2) now becomes

$$\overline{T}_{\mu}{}^{\lambda}{}_{\nu} B^{\mu}_{I)} B^{\nu}_{J)} = \underline{T}_{I}{}^{K}{}_{J} B^{\lambda}_{K)}, \tag{5.1}$$

which is also written in the forms

$$\underline{T}_{I}{}^{K}{}_{J} = \overline{T}_{\mu}{}^{\lambda}{}_{\nu} B^{\mu}_{I)} B^{\nu}_{J)} B^{K)}_{\lambda}, \tag{5.2}$$

$$\overline{T}_{\mu}{}^{\lambda}{}_{\nu} = \underline{T}_{I}{}^{K}{}_{J} B^{I)}_{\mu} B^{J)}_{\nu} B^{\lambda}_{K)}, \tag{5.3}$$

on account of Theorem 2 and Proposition 4.

Similarly, introducing covariant torsion tensor $\underline{T}_*$ with components $\underline{T}_{IJK} = g_{JH}\underline{T}_{I}{}^{H}{}_{K}$, we obtain

$$\underline{T}_{IJK} = \overline{T}_{\lambda\mu\nu} B^{\lambda}_{I)} B^{\mu}_{J)} B^{\nu}_{K)}, \tag{5.4}$$

$$\overline{T}_{\lambda\mu\nu} = \underline{T}_{IJK} B^{I)}_{\lambda} B^{J)}_{\mu} B^{K)}_{\nu}. \tag{5.5}$$

Consider (5.5) in terms of N-decompositions. It is observed that, for instance, $(\overline{T}_{*122})_{ijk} = \underline{T}_{IJK} C^{I)}_{i} D^{J)}_{j} D^{K)}_{k}$. From (2.13) it follows that $\underline{T}_{IJK} C^{I)}_{i} D^{J)}_{j} D^{K)}_{k} = \underline{T}_*(N_{i)}, V_{j)}, V_{k)})$. Therefore (1.5) yields $\underline{T}_*(N_{i)}, V_{j)}, V_{k)}) = P_{ijk}$. Similarly we obtain

$$\begin{aligned} &(1) \quad \underline{T}_*(N_{i)}, N_{j)}, N_{k)}) = \underline{T}_*(V_{i)}, N_{j)}, V_{k)}) = \underline{T}_*(V_{i)}, V_{j)}, V_{k)}) = 0, \\ &(2) \quad \underline{T}_*(N_{i)}, V_{j)}, N_{k)}) = R_{ijk}, \qquad \underline{T}_*(N_{i)}, V_{j)}, V_{k)}) = P_{ijk}, \\ &\qquad \underline{T}_*(N_{i)}, N_{j)}, V_{k)}) = C_{ijk}. \end{aligned} \tag{5.6}$$

By using notations N_y and V_y the above results are expressed in the following invariant form:

Theorem 5. *Values of the covariant torsion tensor $\underline{T}_*$ of indicatrix bundle $I(F^n)$ in the directions $N\times V\times N$, $N\times V\times V$ and $N\times N\times V$ are respectively equal to the covariant (v)h-, (v)hv- and (h)hv-torsion tensors R^1_*, P^1_*, C_* of F^n. On the other hand values in $N\times N\times N$, $V\times N\times V$ and $V\times V\times V$ vanish.*

We turn our discussion to integrability conditions of (4.1) and (4.3). The conditions are nothing but the Gauss and Codazzi equations:

$$\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu} B^{\varkappa}_{H)} B^{I)}_{\lambda} B^{\mu}_{J)} B^{\nu}_{K)} = \underline{R}_{H}{}^{I}{}_{JK} - (H_{HJ}H^{I}{}_{K} - H_{HK}H^{I}{}_{J})(= Q_{H}{}^{I}{}_{JK}), \tag{5.7}$$

$$\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu} B^{\varkappa}_{I)} B_{\lambda} B^{\mu}_{J)} B^{\nu}_{K)} = H_{IJ;K} - H_{IK;J} + H_{IH}\underline{T}_{J}{}^{H}{}_{K}(= Q_{IJK}), \tag{5.8}$$

where $\underline{R}_{H}{}^{I}{}_{JK}$ are components of the curvature tensor of $I(F^n)$ and (;) denotes covariant differentiation with respect to the induced connection $\underline{\Gamma}$. (We put right-hand sides of (5.7) and (5.8) as $Q_{H}{}^{I}{}_{JK}$ and Q_{IJK} respectively).

Pay attention to (5.7). The curvature tensor $\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu}$ may be changed for indicatorized $'\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu}$ because of Proposition 4:

$$'\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu} B^{\varkappa}_{H)} B^{I)}_{\lambda} B^{\mu}_{J)} B^{\nu}_{K)} = Q_{H}{}^{I}{}_{JK}. \tag{5.7'}$$

Consequently we also have

(5.9) $$'\overline{R}_{\varkappa}{}^{\lambda}{}_{\mu\nu} = Q_H{}^I{}_{JK} B^{H)}_{\varkappa} B^{\lambda}_{I)} B^{J)}_{\mu} B^{K)}_{\nu}, \quad '\overline{R}_{\varkappa\lambda\mu\nu} = Q_{HIJK} B^{H)}_{\varkappa} B^{I)}_{\lambda} B^{J)}_{\mu} B^{K)}_{\nu},$$

where $Q_{HIJK}=g_{IL}Q_H{}^L{}_{JK}$ and $'\overline{R}_{\varkappa\lambda\mu\nu}=\bar{g}_{\lambda\sigma}{}'\overline{R}_{\varkappa}{}^{\sigma}{}_{\mu\nu}$.

According to the rule of indicatorization (Proposition 3), (1.6) yields

(5.10) $$\begin{aligned} &('\overline{R}_{1111}, '\overline{R}_{1112}, '\overline{R}_{1122}, '\overline{R}_{2222}) = (R^2_*, P^2_*, S^2_*, S^2_*) \\ &('\overline{R}_{1121})_{hijk} = -('\overline{R}_{1112})_{hikj}, \qquad ('\overline{R}_{2221})_{hijk} = -('\overline{R}_{2212})_{hikj}, \\ &('\overline{R}_{2211})_{hijk} = R_{hijk} - l_h R_{ijk} + l_i R_{hjk}, \\ &('\overline{R}_{2212})_{hijk} = P_{hijk} - l_h P_{ijk} + l_i P_{hjk}, \\ &'\overline{R}_{\xi\eta\zeta\tau} = 0, \quad \xi \neq \eta. \end{aligned}$$

Then, similarly to the case of torsion, we obtain

(5.11) $$\begin{aligned} &Q(N_{h)}, N_{i)}, N_{j)}, N_{k)}) = R_{hijk}, \qquad Q(N_{h)}, N_{i)}, N_{j)}, V_{k)}) = P_{hijk}, \\ &Q(N_{h)}, N_{i)}, V_{j)}, V_{k)}) = Q(V_{h)}, V_{i)}, V_{j)}, V_{k)}) = S_{hijk}, \\ &Q(V_{h)}, V_{i)}, \dot{N}_{j)}, N_{k)}) = R_{hijk} - l_h R_{ijk} + l_i R_{hjk}, \\ &Q(V_{h)}, V_{i)}, N_{j)}, V_{k)}) = P_{hijk} - l_h P_{ijk} + l_i R_{hjk}, \\ &Q(N_{h)}, V_{i)}, *, *) = Q(V_{h)}, N_{i)}, *, *) = 0, \end{aligned}$$

where (*) indicates N or V.

Next we treat left-hand sides of (5.11). To do so, we put $H_{\lambda\mu}=H_{IJ}B^{I)}_{\lambda}B^{J)}_{\mu}$ and find N-decompositions of $H_{\lambda\mu}$. From (4.5) and (2.10′)

$$\begin{aligned} (H_{11})_{ij} &= H_{IJ} C^{I)}_i C^{J)}_j = -h_{hk} D^h_{I)} D^k_{J)} C^{I)}_i C^{J)}_j = 0, \\ (H_{22})_{ij} &= H_{IJ} D^{I)}_i D^{J)}_j = -h_{hk} D^h_{I)} D^k_{J)} D^{I)}_i D^{J)}_j = -h_{ij}. \end{aligned}$$

Similarly we get

(5.12) $$(H_{11}, H_{12}, H_{22}) = (0, 0, -h).$$

Therefore Q in (5.11) may be changed for $\underline{R}_*$, except $Q(V_{h)}, V_{i)}, V_{j)}, V_{k)})=S_{hijk}$, which is equal to $R_*(V_{h)}, V_{i)}, V_{j)}, V_{k)})-h_{hj}h_{ik}+h_{hk}h_{ij}$. Consequently we obtain

(5.13) $$\begin{cases} \underline{R}_*(N_{h)}, N_{i)}, N_{j)}, N_{k)}) = R_{hijk}, \\ \underline{R}_*(N_{h)}, N_{i)}, N_{j)}, V_{k)}) = P_{hijk}, \\ \underline{R}_*(N_{h)}, N_{i)}, V_{j)}, V_{k)}) = S_{hijk}, \end{cases}$$

(5.14) $$\begin{cases} \underline{R}_*(V_{h)}, V_{i)}, N_{j)}, N_{k)}) = R_{hijk} - l_h R_{ijk} + l_i R_{hjk}, \\ \underline{R}_*(V_{h)}, V_{i)}, N_{j)}, V_{k)}) = P_{hijk} - l_h P_{ijk} + l_i P_{hjk}, \\ \underline{R}_*(V_{h)}, V_{i)}, V_{j)}, V_{k)}) = S_{hijk} + (h_{hj}h_{ik} - h_{hk}h_{ij}), \end{cases}$$

(5.15) $$\underline{R}_*(N_{h)}, V_{i)}, *, *) = \underline{R}_*(V_{h)}, N_{i)}, *, *) = 0.$$

Thus we have

Theorem 6. *Values of the covariant curvature tensor* $\underline{R}_*$ *of indicatrix bundle* $I(F^n)$ *in the directions* $N\times N\times N\times N$, $N\times N\times N\times V$ *and* $N\times N\times V\times V$ *are respectively equal to the covariant h-, hv- and v-curvature tensors* R^2_*, P^2_*, S^2_* *of* F^n. *Values in* $V\times V\times N\times N$, $V\times V\times N\times V$ *and* $V\times V\times V\times V$ *are respectively equal to tensors* $R_{hijk}-l_h R_{ijk}+l_i R_{hjk}$, $P_{hijk}-l_h P_{ijk}+l_i P_{hjk}$ *and* $S_{hijk}+(h_{hj}h_{ik}-h_{hk}h_{ij})$. *On the other hand values in* $N\times V\times *\times *$ *and* $V\times N\times *\times *$ *vanish.*

Remark. Every equation of (5.14) is noteworthy. In fact,

$$'P_{hijk} = P_{hijk}-l_h P_{ijk}+l_i P_{hjk}$$

is nothing but the *indicatorized tensor of* P_{hijk} in the sense of F^n (cf. the end of *§ 31).

Next the tensor $S_{hijk}+(h_{hj}h_{ik}-h_{hk}h_{ij})$ is interesting in the viewpoint of the concept "*S3-likeness*" (cf. [5] and *Definition 31.4). From *Theorem 31.6 it is seen that if the indicatrix I_x is flat, this tensor vanishes. We have an explicit example of such a space; an n-dimensional Finsler space equipped with a *Berwald—Moór* metric $L=(y^1y^2\dots y^n)^{1/n}$ [8].

Finally the tensor $R_{hijk}-l_h R_{ijk}+l_i R_{hjk}$ is rather strange. In fact, this is not the indicatorized tensor $'R_{hijk}$ of R_{hijk}:

$$'R_{hijk} = (R_{hijk}-l_h R_{ijk}+l_i R_{hjk})+l_j R_{hiko}-l_k R_{hijo}-$$

$$-l_h l_j R_{iko}-l_i l_k R_{hjo}+l_h l_k R_{ijo}+l_i l_j R_{hko}.$$

In case of P_{hijk}, we have $P_{hijo}=P_{hio}=0$.

Now we are concerned with the Codazzi equation (5.8). If we put $Q_{\lambda\mu\nu}= =Q_{IJK}B^{I)}_\lambda B^{J)}_\mu B^{K)}_\nu$, the left-hand side of (5.8) gives

$$Q_{\lambda\mu\nu} = \overline{R}_\lambda{}^\varkappa{}_{\mu\nu}B_\varkappa - B^\sigma B_\varkappa \overline{R}_\sigma{}^\varkappa{}_{\mu\nu}B_\lambda.$$

Since $C\overline{\Gamma}$ is metrical, $\overline{R}_{\sigma\varkappa\mu\nu}$ is skew-symmetric in σ and $\varkappa$, so that the second term of the right-hand side of the above vanishes. Then, putting $\overline{R}_{\lambda\mu\nu}=-\overline{R}_\lambda{}^\varkappa{}_{\mu\nu}B_\varkappa$, we get $Q_{\lambda\mu\nu}=-\overline{R}_{\lambda\mu\nu}$. We shall find N-decompositions of $\overline{R}_{\lambda\mu\nu}$. For instance, from (1.6) and (2.5′) we have

$$-(\overline{R}_{211})_{ijk} = (\overline{R}_2{}^1{}_{11})_i{}^h{}_{jk}C_{*h}+(\overline{R}_2{}^2{}_{11})_i{}^h{}_{jk}D_{*h} = -R_{ijk}.$$

Similarly we have

$$(\overline{R}_{111}, \overline{R}_{211}, \overline{R}_{112}, \overline{R}_{212}, \overline{R}_{122}, \overline{R}_{222}) = (0, R^1_*, 0, P^1_*, 0, 0). \tag{5.16}$$

On the other hand, we put $U_{\lambda\mu\nu}=(H_{IH}\underline{T}_J{}^H{}_K)B^{I)}_\lambda B^{J)}_\mu B^{K)}_\nu$, which is equal to $H_{\lambda\varkappa}\overline{T}_\mu{}^\varkappa{}_\nu$, as it is easily verified from Theorem 2. From (1.5) and (5.12) it is seen that N-decompositions of $U_{\lambda\mu\nu}$ are given by $(U_{111}, U_{211}, U_{112}, U_{212}, U_{122}, U_{222})= =(0, R^1_*, 0, P^1_*, 0, 0)$, so that $U_{\lambda\mu\nu}=\overline{R}_{\lambda\mu\nu}$ from (5.16):

$$H_{IH}\underline{T}_J{}^H{}_K = -\overline{R}_\varkappa{}^\lambda{}_{\mu\nu}B^\varkappa_{I)}B_\lambda B^\mu_{J)}B^\nu_{K)} \tag{5.17}$$

and (5.8) is written in the form

$$H_{IJ;K} - H_{IK;J} + 2H_{IH}\underline{T}_J{}^H{}_K = 0. \tag{5.18}$$

Summarizing up the above, we have

Theorem 7. *Putting* $H_{IJK} = H_{IJ;K} - H_{IK;J}$, *values of* H_{IJK} *in the directions* $N\times N\times N$, $N\times N\times V$, $N\times V\times V$ *and* $V\times V\times V$ *vanish. On the other hand values* $H(V\times N\times N)$ *and* $H(V\times N\times V)$ *of* H_{IJK} *in* $V\times N\times N$ *and* $V\times N\times V$ *are respectively equal to* $-2R^1_*$ *and* $-2P^1_*$.

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