

On the conditional laws of large numbers II

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1. This paper, which is a continuation of our previous publication [3], is concerned with the strong laws of large numbers for a conditional probability field of RÉNYI. We shall deal with the generalizations of some classical strong laws. The proofs are based on a generalization of Kolmogorov's inequality, and a generalization of the inequality of HAJEK and RÉNYI [2]. Naturally, in the special case of a probability space (in the sense of Kolmogorov) we arrive at some well-known laws of large numbers.

We shall use the same symbols as in [3]. A conditional probability field will be denoted by $[H, T_1, T_2, p]$. Throughout this article we employ some constant notations:

Let $\xi_1, \xi_2, \dots$ be random variables on the conditional probability field $[H, T_1, T_2, p]$ and Q be a Borel set of the real line. Let $C \in T_2$ and let us assume that

$$\xi_n^{-1}(Q) \in T_2, \quad \xi_n^{-1}(Q) \subseteq C \quad (n = 1, 2, \dots).$$

Let $B_n = \xi_n^{-1}(Q)$, $p_n = p(B_n|C)$, $E_n = E(\xi_n|B_n)$, $D_n = D(\xi_n|B_n)$ provided that $E(\xi_n|B_n)$, $D(\xi_n|B_n)$ ($n = 1, 2, \dots$) exist. Finally let

$$\varepsilon_i = \begin{cases} 1, & \xi_i \in Q \\ 0, & \xi_i \notin Q \end{cases}$$

$$\tau_n = \sum_{i=1}^n \varepsilon_i, \quad \text{and} \quad v_n = \sum_{i=1}^n \varepsilon_i (\xi_i - E_i).$$

If the sequence $\xi_1, \xi_2, \dots$ converges to a random variable ξ with probability 1 (with respect to C), it will be denoted by $\xi_n \xrightarrow{C} \xi$.

A more detailed list of notions and notations can be found in [3].

2. The following result is a simple generalization of Kolmogorov's inequality.

Lemma 2.1. *Let $\xi_1, \xi_2, \dots$ be independent with respect to C and assume that the conditional expectations and variances $E_i = E(\xi_i|B_i)$, $D_i = D(\xi_i|B_i)$ ($i = 1, 2, \dots$) exist. If $\omega_1, \omega_2, \dots$ are random variables such that $0 < \omega_1 \leq \omega_2 \leq \dots$ and there exist $E(\omega_i^2|C) > 0$, then for all $\varepsilon > 0$*

$$(2.1) \quad p\left(\max_{1 \leq k \leq n} \left| \frac{v_k}{\omega_k} \right| \geq \varepsilon | C\right) \leq \frac{1}{\varepsilon^2} \sum_{i=1}^n \frac{p_i D_i^2}{E(\omega_i^2|C)}.$$

PROOF. Let n be fixed and $\xi_i^* = \varepsilon_i(\xi_i - E_i)$. Then $v_k = \sum_{i=1}^k \xi_i^*$. Let

$$A_k = \{|v_1| < \omega_1 \varepsilon\} \cap \{|v_2| < \omega_2 \varepsilon\} \cap \dots \cap \{|v_{k-1}| < \omega_{k-1} \varepsilon\} \cap \{|v_k| \equiv \omega_k \varepsilon\}$$

$$(k = 1, 2, 3, \dots, n)$$

and

$$A_0 = \bigcap_{i=1}^n \{|v_i| < \omega_i \varepsilon\}.$$

We need the following computations:

If $k \leq i \leq n$, then $v_i = v_k + \sum_{j=k+1}^i \xi_j^*$ and

$$(2.2) \quad v_i^2 = v_k^2 + \sum_{j=k+1}^i \xi_j^{*2} + 2 \sum_{j=k+1}^i v_k \xi_j^* + 2 \sum_{k < j_1 < j_2 \leq i} \xi_{j_1}^* \xi_{j_2}^*.$$

Let now $h \in H$ and

$$\alpha_k(h) = \begin{cases} 1, & h \in A_k \\ 0, & h \notin A_k. \end{cases}$$

Then, if $k < i < j \leq n$ and $p(A_k|C) \neq 0$

$$\begin{aligned} E(\xi_i^* \xi_j^* | A_k C) &= \int_H \xi_i^* \xi_j^* dp(A|A_k C) = \int_H \xi_i^* \xi_j^* \frac{dp(AA_k|C)}{p(A_k|C)} = \\ &= \frac{1}{p(A_k|C)} \int_H \xi_i^* \xi_j^* \alpha_k dp(A|C) = 0 \end{aligned}$$

because $\xi_i^*, \xi_j^*, \alpha_k$ are independent with respect to C and $E(\xi_i^*|C) = 0$ ($i = 1, 2, \dots, n$).

In a similar way we get

$$E(v_k \xi_j^* | A_k C) = 0 \quad (k < j \leq n) \quad \text{if} \quad p(A_k|C) \neq 0.$$

Thus by (2.2)

$$(2.3) \quad E(v_i^2 | A_k C) \equiv E(v_i | A_k C) + K_{k,i} \quad (k \leq i < n),$$

where

$$K_{k,i} = E\left(2 \left[\sum_{j=k+1}^i v_k \xi_j^* + \sum_{k < j_1 < j_2 \leq i} \xi_{j_1}^* \xi_{j_2}^* \right] | A_k C\right),$$

and

$$K_{k,i} = 0 \quad \text{if} \quad p(A_k|C) \neq 0.$$

Let

$$\eta_n = \sum_{k=1}^{n-1} v_k^2 \left(\frac{1}{E(\omega_k^2|C)} - \frac{1}{E(\omega_{k+1}^2|C)} \right) + \frac{v_n^2}{E(\omega_n^2|C)}.$$

Since

$$E(v_i^2|C) = \sum_{j=1}^i p_j D_j^2$$

$$(2.4) \quad E(\eta_n|C) = \sum_{i=1}^n \frac{p_i D_i^2}{E(\omega_i^2|C)}.$$

By (2.3) it follows that

$$\begin{aligned}
 E(\eta_n|A_k C) &= \sum_{i=1}^{n-1} E(v_i^2|A_k C) \left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)} \right) + \frac{E(v_n^2|A_k C)}{E(\omega_n^2|C)} \equiv \\
 &\equiv \sum_{i=k}^{n-1} E(v_i^2|A_k C) \left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)} \right) + \frac{E(v_n^2|A_k C)}{E(\omega_n^2|C)} \equiv \\
 &\equiv \sum_{i=k}^{n-1} E(v_i^2|A_k C) \left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)} \right) + \frac{E(v_k^2|A_k C)}{E(\omega_k^2|C)} + \\
 &+ \sum_{i=k}^{n-1} K_{k,i} \left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)} \right) + \frac{K_{k,n}}{E(\omega_n^2|C)} = \\
 &= \frac{E(v_k^2|A_k C)}{E(\omega_k^2|C)} + L_k \equiv \varepsilon^2 \frac{E(\omega_k^2|C)}{E(\omega_k^2|C)} + L_k = \varepsilon^2 + L_k \quad (k = 1, 2, \dots, n),
 \end{aligned}$$

where $L_k=0$ if $p(A_k|C) \neq 0$.

Hence

$$(2.5) \quad E(\eta_n|A_k C) \equiv \varepsilon^2 + L_k \quad (k = 1, 2, \dots, n),$$

where $L_k=0$ if $p(A_k|C) \neq 0$.

Since $\{A_0, A_1, \dots, A_n\}$ is a complete group of events

$$E(\eta_n|C) \equiv \sum_{k=1}^n E(\eta_n|A_k C) p(A_k|C)$$

which implies

$$\begin{aligned}
 E(\eta_n|C) &\equiv \sum_{k=1}^n \varepsilon^2 p(A_k|C) + \sum_{k=1}^n L_k p(A_k|C) = \\
 &= \varepsilon^2 \sum_{k=1}^n p(A_k|C) = \varepsilon^2 p\left(\bigcup_{i=1}^n A_i|C\right).
 \end{aligned}$$

Hence by (2.4)

$$\varepsilon^2 p\left(\bigcup_{i=1}^n A_i|C\right) \equiv \sum_{i=1}^n \frac{p_i D_i^2}{E(\omega_i^2|C)},$$

which proves (2.1).

Corollary. If $\omega_k = \sum_{i=1}^k \varepsilon_i$, then Lemma 2.1 means that

$$(2.6) \quad p\left(\max_{1 \leq k \leq n} \left| \frac{\sum_{i=1}^k \varepsilon_i (\xi_i - E_i)}{\sum_{i=1}^k \varepsilon_i} \right| \geq \varepsilon | C\right) \equiv \frac{1}{\varepsilon^2} \sum_{i=1}^n \frac{p_i D_i^2}{\left(\sum_{j=1}^i p_j\right)^2}$$

because of $E\left(\left(\sum_{j=1}^i \varepsilon_j\right)^2 | C\right) \equiv \left(\sum_{j=1}^i p_j\right)^2$ by Lemma 3.1 of [3].

Our following two theorems (Theorem 2.2 and Theorem 2.3) are analogous to a classical result of Kolmogorov [6, Theorem 2.7.5a]. A similar result was obtained by RÉNYI who proved the following (see [4, pp. 401]):

If $\xi_1, \xi_2, \dots$ are independent with respect to C with finite expectation and variance $E_k = E(\xi_k | B_k)$, $D_k = D(\xi_k | B_k)$ ($k = 1, 2, \dots$), and

$$\begin{aligned} \text{a)} \quad & \sum_{k=1}^{\infty} p_k = \infty, \\ \text{b)} \quad & \sum_{k=1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^k p_j \right)^2} < \infty, \end{aligned}$$

then

$$\frac{\sum_{i=1}^k \varepsilon_i \xi_i}{\sum_{i=1}^k \varepsilon_i} \xrightarrow{C} E \quad (k \rightarrow \infty).$$

Our theorems will be proved under somewhat different conditions and by a totally different method.

Theorem 2.2. *Let the random variables $\xi_1, \xi_2, \dots$ be independent with respect to C and suppose that E_i, D_i exist ($i = 1, 2, \dots$). Moreover let us assume that*

$$1) \quad \sum_{k=1}^{\infty} \frac{D_k^2}{k^2} < \infty, \quad 2) \quad 0 < d \leq p_i \quad (i = 1, 2, \dots).$$

Then

$$\frac{v_k}{\tau_k} = \frac{\sum_{i=1}^k \varepsilon_i (\xi_i - E_i)}{\sum_{i=1}^k \varepsilon_i} \xrightarrow{C} 0 \quad (k \rightarrow \infty).$$

PROOF. Let

$$\delta_N = \sup_{n \geq N} \left| \frac{v_n}{\tau_n} \right|, \quad \delta_{a,b} = \max_{a \leq n \leq b} \left| \frac{v_n}{\tau_n} \right|.$$

If $\delta_N > \varepsilon$ and $2^s \leq N < 2^{s+1}$ for an integer s , then there exists $l \geq s$ such that $\delta_{2^l 2^{l+1}} \geq \frac{\varepsilon}{2}$, and so

$$p(\delta_N \geq \varepsilon | C) \leq \sum_{l=s}^{\infty} p\left(\delta_{2^l 2^{l+1}} \geq \frac{\varepsilon}{2} \middle| C\right) \quad \text{if } 2^s \leq N < 2^{s+1}.$$

Furthermore

$$\begin{aligned}
 p(\delta_{2^l, 2^{l+1}} \equiv \varepsilon | C) &= p\left(\max_{2^l \leq k < 2^{l+1}} \left| \frac{v_k}{\tau_k} \right| \equiv \frac{\varepsilon}{2} | C\right) \equiv p\left(\max_{2^l \leq k < 2^{l+1}} \sum_{i=1}^k p_i \left| \frac{v_k}{\tau_k} \right| \equiv \right. \\
 &\equiv \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i | C) = p\left(\max_{2^l \leq k < 2^{l+1}} \left| \frac{v_k}{\tau'_k} \right| \equiv \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i | C\right) \equiv \\
 &\equiv p\left(\max_{1 \leq k < 2^{l+1}} \left| \frac{v_k}{\tau'_k} \right| \equiv \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i | C\right) \equiv \frac{4}{\varepsilon^2} \left(\sum_{k=1}^{2^l} p_k\right)^{-2} \sum_{k=1}^{2^{l+1}-1} \frac{p_k D_k^2}{E(\tau_k'^2 | C)} \equiv \\
 &\equiv \frac{4}{\varepsilon^2} \left(\sum_{k=1}^{2^l} p_k\right)^{-2} \sum_{k=1}^{2^{l+1}-1} p_k D_k^2,
 \end{aligned}$$

where $\tau'_k = \frac{\tau_k}{\sum_{i=1}^k p_i}$. Hence

$$\begin{aligned}
 p(\delta_N \equiv \varepsilon | C) &\equiv \frac{4}{\varepsilon^2} \sum_{l=s}^{\infty} \frac{\sum_{k=1}^{2^{l+1}-1} p_k D_k^2}{\left(\sum_{k=1}^{2^l} p_k\right)^2} \equiv \frac{4}{\varepsilon^2 d^2} \sum_{l=s}^{\infty} \frac{1}{2^{2l}} \sum_{k=1}^{2^{l+1}-1} D_k^2 \equiv \\
 &\equiv \frac{4}{\varepsilon^2 d^2} \left(\frac{1}{4^{s-1}} \sum_{k=1}^{2^{s+1}-1} D_k^2 + 16 \sum_{k=2^{s+1}}^{\infty} \frac{D_k^2}{k^2} \right).
 \end{aligned}$$

Now by the Kronecker lemma $\frac{\sum_{i=1}^n D_i^2}{n^2} \rightarrow 0$ ($n \rightarrow \infty$), consequently the last expression

converges to zero which implies our theorem.

The classical form of the following lemma is due to HAJEK and RÉNYI [2].

Lemma 2.3. *Let us assume that the conditional expectations and variances of $\xi_1, \xi_2, \dots$ exist, i.e. $E_i = E(\xi_i | B_i) < \infty$, $D_i = D(\xi_i | B_i) < \infty$, and*

$$\sum_{i=1}^{\infty} p_i = \infty, \quad \sum_{i=1}^{\infty} \frac{p_i D_i^2}{\left(\sum_{j=1}^i p_j\right)^2} < \infty.$$

Then for every $\varepsilon > 0$ and $\sum_{i=1}^n p_i > 0$

$$(2.7) \quad p\left(\sup_{k \geq n} \left| \frac{v_k}{\tau_k} \right| \equiv \varepsilon | C\right) \equiv \frac{1}{\varepsilon} \left(\frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2 | C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2 | C)} \right).$$

PROOF. First of all we remark that the right hand side of equation (2.7) is defined because by Lemma 3.1 of [3] we have

$$(2.8) \quad E(\tau_n^2|C) \cong \left(\sum_{i=1}^n p_i \right)^2 \quad (n = 1, 2, \dots).$$

We need some calculations. Let $\xi_i^* = \varepsilon_i(\xi_i - E_i)$, then

$$\begin{aligned} E(v_k^2|C) &= E\left(\left(\sum_{i=1}^k \xi_i^*\right)^2 \middle| C\right) = \sum_{i=1}^k \int_{B_i} (\xi_i - E_i)^2 dP(A|C) + \\ &+ 2 \sum_{1 \leq i < j \leq k} E(\xi_i^*, \xi_j^*|C) = \sum_{i=1}^k \int_{B_i} (\xi_i - E_i)^2 p_i dP(A|B_i) = \sum_{i=1}^k p_i D_i^2 \end{aligned}$$

because $E(\xi_i^*|C) = 0$ ($i = 1, 2, \dots$) and ξ_i^*, ξ_j^* are independent with respect to C . Hence

$$(2.9) \quad E(v_k^2|C) = \sum_{i=1}^k p_i D_i^2.$$

Let now

$$\eta_n = \sum_{k=n}^{\infty} v_k^2 \left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)} \right).$$

Then

$$\begin{aligned} E(\eta_n|C) &= \frac{E(v_n^2|C)}{E(\tau_n^2|C)} - \frac{E(v_n^2|C)}{E(\tau_{n+1}^2|C)} + \frac{E(v_{n+1}^2|C)}{E(\tau_{n+1}^2|C)} - \frac{E(v_{n+1}^2|C)}{E(\tau_{n+2}^2|C)} + (-) \dots = \\ &= \frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2|C)} - \frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_{n+1}^2|C)} + \frac{\sum_{i=1}^{n+1} p_i D_i^2}{E(\tau_{n+1}^2|C)} - \frac{\sum_{i=1}^{n+1} p_i D_i^2}{E(\tau_{n+2}^2|C)} + (-) \dots = \\ &= \frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)} \end{aligned}$$

and these series are convergent by the Kronecker lemma and our conditions. Thus

$$(2.10) \quad E(\eta_n|C) = \frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)}.$$

Let us denote the events A_m ($m = n, n+1, \dots$) by

$$A_m = \left\{ \left| \frac{v_n}{\tau_n} \right| < \varepsilon, \left| \frac{v_{n+1}}{\tau_{n+1}} \right| < \varepsilon, \dots, \left| \frac{v_{m-1}}{\tau_{m-1}} \right| < \varepsilon, \left| \frac{v_m}{\tau_m} \right| \geq \varepsilon \right\}.$$

Since $E(v_m^2|A_m C) \cong E(\tau_m^2 \varepsilon^2|A_m C) \cong E(\tau_m^2 \varepsilon^2|C) = \varepsilon^2 E(\tau_m^2|C)$ we can state

$$(2.11) \quad E(v_m^2|A_m C) \cong \varepsilon^2 E(\tau_m^2|C) \quad (m = n, n+1, \dots).$$

On the other hand, if $k \geq m$ then

$$\begin{aligned} E(v_k^2 | A_m C) &= E(v_m^2 + 2v_m(v_k - v_m) + (v_k - v_m)^2 | A_m C) \cong \\ &\cong E(v_m^2 | A_m C) + E(2v_m(v_k - v_m) | A_m C), \end{aligned}$$

where

$$v_m = \sum_{i=1}^m \varepsilon_i(\xi_i - E_i) \quad \text{and} \quad v_k - v_m = \sum_{i=m+1}^k \varepsilon_i(\xi_i - E_i)$$

are independent with respect to C .

Let

$$\alpha_m(a) = \begin{cases} 1, & a \in A_m \\ 0, & a \notin A_m. \end{cases}$$

Then $v_k - v_m$ and α_m are also independent with respect to C , so if $P(A_m | C) \neq 0$ we get

$$\begin{aligned} E(v_m(v_k - v_m) | A_m C) &= \int_H v_m(v_k - v_m) dP(A | A_m C) = \\ &= \int_H \frac{v_m(v_k - v_m)}{P(A_m | C)} dP(A A_m | C) = \int_H \frac{v_m(v_k - v_m)}{P(A_m | C)} \alpha_m dP(A | C) = \\ &= \frac{1}{P(A_m | C)} E(v_m \alpha_m | C) E(v_k - v_m | C) = 0. \end{aligned}$$

Denoting $K_{m,k}$ ($m \leq k$ by $K_{m,k} = E(2v_m(v_k - v_m) | A_m C)$) we have

$$(2.12) \quad K_{m,k} = \begin{cases} 2 \sum_{1 \leq i < j \leq k} E(\xi_i^* \xi_j^* | A_m C) & \text{if } m < k \text{ and } P(A_m | C) = 0 \\ 0 & \text{if } m = k \text{ or } P(A_m | C) \neq 0. \end{cases}$$

Hence

$$E(v_k^2 | A_m C) \cong E(v_m^2 | A_m C) + K_{m,k} \quad (k \geq m)$$

so with the help of (2.11)

$$\begin{aligned} E(\eta_n | A_m C) &= \sum_{k=n}^{\infty} E(v_k^2 | A_m C) \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right) \cong \\ &\cong \sum_{k=m}^{\infty} E(v_m^2 | A_m C) \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right) + \sum_{k=m}^{\infty} K_{m,k} \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right) \cong \\ &\cong \sum_{k=m}^{\infty} \varepsilon^2 E(\tau_m^2 | C) \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right) + \sum_{k=m}^{\infty} K_{m,k} \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right) = \\ &= \varepsilon^2 + \sum_{k=m}^{\infty} K_{m,k} \left(\frac{1}{E(\tau_k^2 | C)} - \frac{1}{E(\tau_{k+1}^2 | C)} \right). \end{aligned}$$

Let

$$L_m = \sum_{k=m}^{\infty} K_{m,k} \left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)} \right),$$

then

$$(2.13) \quad E(\eta_n|A_m C) \cong \varepsilon^2 + L_m \quad (m = n, n+1, \dots).$$

Now it is evident by (2.12) that

$$(2.14) \quad L_m = 0 \quad \text{if} \quad P(A_m|C) \neq 0.$$

Let $A_0 = C \setminus \bigcup_{i=n}^{\infty} A_i$. Then $E_0, E_n, E_{n+1}, \dots$ is a complete group of events, so there follows

$$\begin{aligned} E(\eta_n|C) &= \sum_{m=n}^{\infty} E(\eta_n|A_m C) P(A_m|C) + E(\eta_n|A_0 C) P(A_0|C) \cong \\ &\cong \sum_{m=n}^{\infty} E(\eta_n|A_m C) P(A_m|C) \cong \sum_{m=n}^{\infty} (\varepsilon^2 + L_m) P(A_m|C) = \\ &= \varepsilon^2 \sum_{m=n}^{\infty} P(A_m|C) + \sum_{m=n}^{\infty} L_m P(A_m|C) = \\ &= \varepsilon^2 P\left(\bigcup_{m=n}^{\infty} A_m|C\right) = \varepsilon^2 P\left(\sup_{k \geq n} \left| \frac{v_k}{\tau_k} \right| \cong \varepsilon|C\right). \end{aligned}$$

Hence

$$(2.15) \quad E(\eta_n|C) \cong \varepsilon^2 P\left(\sup_{k \geq n} \left| \frac{v_k}{\tau_k} \right| \cong \varepsilon|C\right).$$

Finally, by (2.10) and (2.15) our statement follows.

As a straightforward consequence of the previous lemma, we can now state the following result.

Theorem 2.4. *Let $\xi_1, \xi_2, \dots$ be independent with respect to C and assume that $E_k = E(\xi_k|B_k)$, $D_k = D(\xi_k|B_k)$ exist. We also assume that*

$$\text{a) } \sum_{k=1}^{\infty} p_k = \infty$$

and

$$\text{b) } \sum_{k=1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^k p_j\right)^2} < \infty.$$

Then

$$\frac{v_n}{\tau_n} = \frac{\sum_{i=1}^n \varepsilon_i (\xi_i - E_i)}{\sum_{i=1}^n \varepsilon_i} \xrightarrow{C} 0 \quad (n \rightarrow \infty).$$

PROOF. It is enough to prove that

$$\frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)} \rightarrow 0 \quad (n \rightarrow \infty).$$

But from (2.8)

$$\frac{\sum_{i=1}^n p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)} \equiv \frac{\sum_{i=1}^n p_i D_i^2}{\left(\sum_{i=1}^n p_i\right)^2} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^k p_j\right)^2}$$

and the right side of this inequality tends to zero by the Kronecker lemma.

Remark. Beside the hypotheses of Theorem 2.4, assume that

$$\sum_{n=1}^{\infty} \frac{\sum_{i=1}^n p_i |E_i - E|}{\sum_{i=1}^n p_i} < +\infty.$$

Then the reader can readily verify that

$$\frac{\sum_{i=1}^n \varepsilon_i \zeta_i}{\tau_n} \xrightarrow{C} E \quad (n \rightarrow \infty).$$

In the following three theorems we shall prove that the strong law of large numbers holds not only for mutually independent random variables. If $\xi_1, \xi_2, \dots$ are not assumed to be mutually independent then, of course, we need some stronger hypotheses.

Theorem 2.5. Let $\xi_1, \xi_2, \dots$ be identically distributed random variables and let any four of them be independent with respect to C . Suppose that

$$a) \quad E(\xi_k^4|B_k) = E^{(4)} < \infty, \quad E(\xi_k|B_k) = 0 \quad (k = 1, 2, \dots),$$

$$b) \quad \sum_{i=1}^{\infty} \frac{1}{\left(\sum_{j=1}^i p_j\right)^2} < \infty.$$

Then $\frac{\sum_{i=1}^n \varepsilon_i \zeta_i}{\tau_n} \xrightarrow{C} 0 \quad (n \rightarrow \infty).$

PROOF. Apply Lemma 3.1 and Lemma 3.2 of [3]. Then

$$\begin{aligned}
 P\left(\left|\frac{\sum_{i=1}^n \varepsilon_i \zeta_i}{\tau_n}\right| \geq \varepsilon | C\right) &\leq \frac{1}{\varepsilon^4} \frac{E(v_n^4 | C)}{E(\tau_n^4 | C)} \leq \frac{1}{\varepsilon^4} \frac{E(v_n^4 | C)}{\left(\sum_{j=1}^n p_j\right)^4} = \\
 &= \frac{\sum_{i=1}^n p_i E^{(4)} + 6D^4 \sum_{1 \leq i < j \leq n} p_i p_j}{\varepsilon^4 \left(\sum_{i=1}^n p_i\right)^4} \leq \frac{K \left(\sum_{i=1}^n p_i + 2 \sum_{1 \leq i < j \leq n} p_i p_j\right)}{\left(\sum_{i=1}^n p_i\right)^4} = \\
 &= \frac{K \left(\sum_{i=1}^n p_i\right) + \left(\sum_{i=1}^n p_i\right)^2}{\left(\sum_{i=1}^n p_i\right)^4} \leq \frac{2K' \left(\sum_{i=1}^n p_i\right)^2}{\left(\sum_{i=1}^n p_i\right)^4} = \frac{2K'}{\left(\sum_{i=1}^n p_i\right)^2}
 \end{aligned}$$

where the constants K, K' do not depend on n . Hence

$$\sum_{n=1}^{\infty} P\left(\left|\frac{v_n}{\tau_n}\right| \geq \varepsilon | C\right) \leq 2K' \sum_{i=1}^{\infty} \frac{1}{\left(\sum_{j=1}^i p_j\right)^2} < \infty$$

which implies the theorem.

Theorem 2.6. Let $\zeta_1, \zeta_2, \dots$ be identically distributed random variables with finite conditional expectations and variances

$$E = E(\zeta_i | B_i), \quad D = D(\zeta_i | B_i) \quad (i = 1, 2, \dots).$$

Furthermore, let us suppose that a) $\zeta_1, \zeta_2, \dots$ are pairwise uncorrelated random variables, i.e., $E(\varepsilon_i(\zeta_i - E(\zeta_i | B_i))\varepsilon_j(\zeta_j - E(\zeta_j | B_j)) | C) = 0$ ($i \neq j$), b) $0 < d \leq p_i$ ($i = 1, 2, \dots$). Then

$$\frac{\sum_{i=1}^n \varepsilon_i \zeta_i}{\tau_n} \xrightarrow{C} E \quad (n \rightarrow \infty).$$

PROOF. Let $\zeta_i^* = \varepsilon_i(\zeta_i - E)$ ($i = 1, 2, \dots$). For every positive integer N there exists n such that $n^2 \leq N < (n+1)^2$. Thus

$$\left|\frac{\sum_{i=1}^N \zeta_i^*}{\tau_N}\right| \leq \frac{|\zeta_1^* + \zeta_2^* + \dots + \zeta_{n^2}^*|}{\tau_{n^2}} + \frac{|\zeta_{n^2+1}^* + \dots + \zeta_N^*|}{\tau_{n^2}}.$$

This inequality shows that it is enough to prove that

$$(2.16) \quad \frac{v_{n^2}}{\tau_{n^2}} = \frac{\sum_{i=1}^{n^2} \zeta_i^*}{\tau_{n^2}} \xrightarrow{C} 0 \quad (n \rightarrow \infty)$$

and

$$(2.17) \quad \frac{v_N - v_{n^2}}{\tau_{n^2}} \xrightarrow{C} 0 \quad (n \rightarrow \infty).$$

By Lemma 3.1 and Lemma 3.2 of [3]

$$P\left(\left|\frac{v_{n^2}}{\tau_{n^2}}\right| \equiv \varepsilon | C\right) \equiv \frac{1}{\varepsilon^2} \frac{E(v_{n^2}^2 | C)}{E(\tau_{n^2} | C)} \equiv \frac{D^2 \sum_{i=1}^{n^2} p_i}{\left(\sum_{i=1}^{n^2} p_i\right)^2} = \frac{D^2}{\sum_{i=1}^{n^2} p_i}.$$

Hence

$$\sum_{n=1}^{\infty} P\left(\left|\frac{v_{n^2}}{\tau_{n^2}}\right| \equiv \varepsilon | C\right) \equiv D^2 \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n^2} p_i} \equiv \frac{D^2}{d} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

which implies that (2.16) holds.

On the other hand

$$P\left(\max_{n^2 \leq N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right) \equiv P\left(\frac{|v_N - v_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right)$$

thus it is sufficient to show for (2.17) that

$$P\left(\max_{n^2 \leq N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right) \rightarrow 1 \quad (n \rightarrow \infty).$$

But this last condition is fulfilled if

$$(2.18) \quad \sum_{n=1}^{\infty} P\left(\max_{n^2 \leq N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C\right) < \infty.$$

By Lemma 3.2 of [3]

$$\begin{aligned} P\left(\max_{n^2 \leq N < (n+1)^2} \frac{\left|\sum_{i=n^2+1}^N \zeta_i^*\right|}{\tau_{n^2}} \equiv \varepsilon | C\right) &\equiv \sum_{N=n^2}^{(n+1)^2-1} P\left(\frac{\left|\sum_{i=n^2+1}^N \zeta_i^*\right|}{\tau_{n^2}} \equiv \varepsilon | C\right) \equiv \\ &\equiv \sum_{N=n^2}^{(n+1)^2-1} \frac{1}{\varepsilon^2} D^2 \frac{\sum_{i=n^2+1}^N p_i}{\left(\sum_{i=1}^{n^2} p_i\right)^2} \equiv \frac{D^2}{\varepsilon^2} \sum_{N=n^2}^{(n+1)^2-1} \frac{N - n^2}{d^2 n^4} \equiv \frac{D^2}{\varepsilon^2 d^2} \sum_{N=n^2+1}^{(n+1)^2-1} \frac{2n}{n^4} = \frac{4D^2}{\varepsilon^2 d^2} \frac{1}{n^2}. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} P\left(\max_{n^2 \leq N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C\right) \leq \frac{4D^2}{\varepsilon^2 d^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

i.e., (2.18) is fulfilled which was to be proved.

Theorem 2.7. Suppose that the following conditions are fulfilled for the random variables $\xi_1, \xi_2, \dots$: a) There exist $E_i = E(\xi_i | B_i)$, $D_i = D(\xi_i | B_i)$ ($i = 1, 2, \dots$), b) $\xi_1, \xi_2, \dots$ are pairwise uncorrelated,

$$c) 0 < d \equiv p_i \quad (i = 1, 2, \dots), \quad d) \sum_{i=1}^{\infty} p_i D_i^2 \left/ \left(\sum_{j=1}^i p_j \right)^2 \right. < \infty.$$

Then

$$\frac{v_n}{\tau_n} = \frac{\sum_{i=1}^n \varepsilon_i (\xi_i - E_i)}{\tau_n} \xrightarrow{c} 0 \quad (n \rightarrow \infty).$$

PROOF. It is sufficient to show that (2.16) and (2.17) hold. Using Lemma 3.2 of [3]

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\frac{|v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C\right) &\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n^2} p_i D_i^2}{\left(\sum_{i=1}^{n^2} p_i\right)^2} = \frac{p_1 D_1^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\left(\sum_{i=1}^{n^2} p_i\right)^2} + \\ &+ \frac{1}{\varepsilon^2} \sum_{m=2}^{\infty} p_m D_m^2 \sum_{n=m}^{\infty} \left(\sum_{i=1}^{n^2} p_i\right)^{-2} \end{aligned}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2} < 1$ and $0 < d \equiv p_i$ ($i = 1, 2, \dots$) we have

$$\sum_{n=m}^{\infty} \left(\sum_{i=1}^{n^2} p_i\right)^{-2} \leq \frac{1}{d^2} \left(\frac{1}{m^4} + \frac{1}{(m+1)^4} + \dots\right) \leq \frac{1}{d^2} \frac{1}{m^2} \leq \frac{1}{d^2} \frac{1}{m^{3/2}} \leq \frac{1}{d^2} \frac{1}{(p_1 + \dots + p_m)^{3/2}}.$$

Hence

$$\sum_{n=1}^{\infty} P\left(\frac{|v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C\right) \leq K + \frac{1}{\varepsilon^2 d^2} \sum_{m=2}^{\infty} \frac{p_m D_m^2}{\left(\sum_{i=1}^m p_i\right)^{3/2}} < \infty,$$

i.e. (2.16) holds.

For (2.17) it is enough to prove that

$$\sum_{n=1}^{\infty} P\left(\max_{n^2 < N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C\right) < \infty.$$

But

$$\begin{aligned}
 \sum_{n=1}^{\infty} P \left(\max_{n^2 < N < (n+1)^2} \frac{|v_N - v_{n^2}|}{\tau_{n^2}} \equiv \varepsilon | C \right) &\equiv \sum_{n=1}^{\infty} \sum_{N=n^2+1}^{(n+1)^2-1} P \left(\frac{\left| \sum_{i=n^2+1}^N \zeta_i^* \right|}{\tau_{n^2}} \equiv \varepsilon | C \right) \\
 &\equiv \sum_{n=1}^{\infty} \sum_{N=n^2+1}^{(n+1)^2-1} \frac{1}{\varepsilon^2} \frac{\sum_{i=n^2+1}^N p_i D_i^2}{\left(\sum_{i=1}^{n^2} p_i \right)^2} \equiv \sum_{m=1}^{\infty} \frac{2 \sqrt{m} p_m D_m^2}{(p_1 + \dots + p_m)^2} \equiv \\
 &\equiv 2 \sum_{m=1}^{\infty} \frac{\sqrt{p_1 + \dots + p_m} p_m D_m^2}{\sqrt{d} (p_1 + \dots + p_m)^2} = \frac{2}{\sqrt{d}} \sum_{m=1}^{\infty} \frac{p_m D_m^2}{\left(\sum_{i=1}^m p_i \right)^{\frac{3}{2}}} < \infty,
 \end{aligned}$$

which implies (2.17). Thus the theorem is proved.

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