

Multiplier algebras and minimal embeddability

By W. G. LEAVITT (Lincoln, Neb.) and L. C. A. VAN LEEUWEN (Groningen)

1. Introduction

Let K be an algebra over a commutative ring k with 1. We will say that K is *minimally embeddable* (and designate it m.e.) if whenever there is a ring A such that $K \triangleleft A$ then there exists some $B \triangleleft A$ such that either $A = K \oplus B$ or else there is some $z \in A$ such that A is generated by $K \oplus B$ and z , with $A/(K \oplus B) \cong k$. If K has this property but only when A is a k -algebra then we will say K is *algebra minimally embeddable* (designated a.m.e.). Clearly any algebra with unit is an m.e. algebra and in a recent paper [1] simple m.e. algebras without unit were constructed over any $k = \mathbb{Z}_p$. In the present paper we will show that there exist simple a.m.e. algebras without unit over an arbitrary field k . We will give a method for constructing such algebras and show that there are simple m.e. algebras without unit, different from those of [1], including certain algebras of J. C. ROBSON [2].

Our construction begins with any k -algebra R with unit containing a regular element a . Let I be the right ideal of R generated by a , and let T the idealizer of I in R . We show that there exists a certain "algebra of multipliers" which we will call "Mult- I ", and prove that the algebra Mult- I is k -isomorphic with T . It then follows that for any $K \cong I$ imbedded as an ideal of a k -algebra A , the k -monomorphism $K \rightarrow T$ can be lifted via Mult- I to a k -homomorphism $A \rightarrow T$. As a consequence we obtain a criterion for minimal embeddability: A necessary and sufficient condition for K to be an a.m.e. algebra over a field k (or an m.e. algebra if $k = \mathbb{Z}_p$) is that either $T = I$ or $T = I + k$.

2. Multiplier algebras

Let k be a commutative ring with 1 and suppose there exists a k -algebra R with unit containing a regular element a . Write $I = aR$ and let T be the idealizer of I in R , that is $T = \{x \in R \mid xa \in I\}$. Writing $k\text{-lin } I$ for the algebra of all k -module endomorphisms of I we will call a pair $(\alpha', \alpha'') \in (k\text{-lin } I) \oplus (k\text{-lin } I)^{\text{op}}$ a *multiplier* of I if it has the property:

$$(1) \quad x\alpha'(y) = \alpha''(x)y$$

for all $x, y \in I$. Clearly the set of all multipliers of I is a sub- k -algebra of $(k\text{-lin } I) \oplus (k\text{-lin } I)^{\text{op}}$ and we will call it *Mult- I* . For $t \in T$ let L_t and R_t denote left and right multiplication by t , then from associativity in R it is clear that the pair $(L_t, R_t) \in \text{Mult-}I$.

Lemma 1. *The map $\varphi: T \rightarrow \text{Mult-}I$ given by $t \mapsto (L_t, R_t)$ is a k -algebra isomorphism.*

PROOF. Clearly φ is a k -algebra homomorphism and if $t \in \text{Ker } \varphi$ then $L_t(a) = ta = 0$ implies $t = 0$. It thus suffices to show that φ is surjective. Suppose $(\sigma', \sigma'') \in \text{Mult-}I$ and let $\sigma'(a) = af$, $\sigma''(a) = ag$. Then from (1) we have $a\sigma'(a) = \sigma''(a)a$ so $a^2f = aga$. By the regularity of a we get $af = ga$ and thus $g \in T$. Then for any arbitrary $au \in I$ we have $\sigma'(au) = aw$ for some $w \in R$, and so

$$a^2w = a\sigma'(au) = \sigma''(a)au = agau.$$

Thus $\sigma'(au) = aw = gau = L_g(au)$. Also $\sigma''(au) = av$ for some $v \in R$ so

$$ava = \sigma''(au)a = au\sigma'(a) = auaf.$$

Thus $va = uaf = uga$ and so $v = ug$. It follows that $\sigma''(au) = av = aug = R_g(au)$, that is $(\sigma', \sigma'') = (L_g, R_g)$.

3. Minimal embeddability

Theorem 2. *Let $K \triangleleft A$ where K and A are k -algebras. If $K \cong I$ then there exists a k -homomorphism $A \rightarrow T$ extending the monomorphism $K \rightarrow I \subseteq T$.*

PROOF. Let $\Theta: K \rightarrow I$ be the given k -isomorphism. For an arbitrary $z \in A$ the mappings $\sigma' = \Theta L_z \Theta^{-1}$ and $\sigma'' = \Theta R_z \Theta^{-1}$ are k -linear maps of I and by associativity in A the pair $(\sigma', \sigma'') \in \text{Mult-}I$. Clearly the map $A \rightarrow \text{Mult-}I$ given by $z \mapsto (\sigma', \sigma'')$ is a k -homomorphism so $\alpha: A \rightarrow T$ where $\alpha(z) = \varphi^{-1}(\sigma', \sigma'')$ is a k -homomorphism of A into T . Now by definition, if $\sigma''(a) = at$ then $\varphi^{-1}(\sigma', \sigma'') = t$. But $\sigma''(a) = \Theta R_z \Theta^{-1}(a) = \Theta(\Theta^{-1}(a)z)$. Thus if $z \in K$ we have $\sigma''(a) = a\Theta(z)$ so $\alpha(z) = \Theta(z)$ and hence α extends $K \rightarrow I \subseteq T$.

Corollary 3. *Let $K \cong I$ where K and I are Z_p -algebras for some prime p . Then if $K \triangleleft A$ for A an arbitrary ring, there exists a homomorphism $A \rightarrow T$ extending the monomorphism $K \rightarrow I \subseteq T$.*

PROOF. Since K has characteristic p we may regard K as an ideal of the Z_p -algebra A/pA and map $A \rightarrow A/pA \rightarrow T$.

For the case k a field we can now establish a criterion for minimal embeddability.

Theorem 4. *Let k be a field and R any k -algebra with unit containing a regular element a . Write $I = aR$ and let T be the idealizer of I in R . Then all k -algebras $K \cong I$ are algebra minimally embeddable if and only if either $T = I$ or $T = I + k$. Moreover, if $k = Z_p$ for some prime p then for "algebra minimally embeddable" may be substituted "minimally embeddable".*

PROOF. If $T = I$ then any $K \cong I$ contains a unit and so is minimally embeddable. Thus suppose $T = I + k$. Let $K \triangleleft A$ where A is a k -algebra and let $B = \text{Ker } \alpha$ where $\alpha: A \rightarrow T$ is the map defined in the proof of Theorem 2. Clearly $K \cap B = 0$ and $\text{Im } \alpha \cong A/B$ contains the image $\Theta(K) = I$. Thus $A/K \oplus B \cong \text{Im } \alpha/I$. Since $\text{Im } \alpha$ is a k -algebra it follows that either $A = K \oplus B$ or $A/K \oplus B \cong T/I \cong k$.

Note that when $k = Z_p$ then, via the mapping of Corollary 3, $\text{Im } \alpha$ is a k -algebra even when A is an arbitrary ring. Thus in this case K is minimally embeddable.

Conversely, if I is algebra minimally embeddable (or minimally embeddable if $k = Z_p$) then $I \triangleleft T$ and $I \cap B = 0$ (where $B \triangleleft T$) implies $B = 0$. Thus either $T = I$ so I contains a unit or else $T/I \cong k$ so $T = I + k$.

Remark that while the above construction is closed under isomorphisms, it is open as to whether or not this is true of the class of all a.m.e. algebras.

Corollary 5. [1, Cor. 2.] *If K is a simple m.e. algebra and A a ring such that $K \triangleleft A$ and $A/K \cong K$, then $A = K \oplus K$. The same result holds when K is a simple a.m.e. k -algebra and A is a k -algebra.*

Corollary 6. [1, Theorem 1 and 1'.] *Let M be a class of simple rings such that if $k \in M$ then either K has a unit or K is an m.e. Z_p -algebra for some prime p . Let W be the set of all such p , then the upper radical defined by M is hereditary if and only if $Z_p \in M$ for all $p \in W$.*

Note that any such radical for non-empty W is a non-special supernilpotent radical.

4. Examples and applications

Example 1. The simple algebra of [1, Section 2].

Note that [1, Lemmas 1—5] can be replaced by our Theorem 2 and Corollary 3. Also [1, Lemma 6] establishes the sufficient condition of Theorem 4, and applies (as was noted in [1, Remarks 1 and 2]) to R over any field Z_p . It is clear, in fact, that a similar proof can be carried out for R over an arbitrary field. Thus the additional result: *There exists a primitive a.m.e. algebra without unit over any field k .*

Example 2. The skew polynomial algebra of [2]. This is the ring $(x-1)S$ where S is the ring of non-commutative polynomials in x and x^{-1} over a field F subject to $fx = xf^\sigma$ for all $f \in F$, where F is the field of rational functions in indeterminates $\{t_n | n \in \mathbb{Z}\}$ over a field k in which σ is the automorphism defined by $t_n^\sigma = t_{n+1}$.

It is easy to see that $x-1$ is regular in S so that Theorem 2 or Corollary 3 applies to the k -algebra $(x-1)S$. Now to show that Theorem 4 applies let $g \in T$ so that $g(x-1) = (x-1)f$ for some $f \in S$. It is easy to check that any $g \in S$ can be written

$$g = (x-1)\mu + g'$$

for some $\mu \in S$ and $g' \in F$. But then $g'(x-1) = (x-1)r$ for some $r \in S$ where (by a degree argument) $r \in F$. Then $xg'^\sigma - g' = xr - r$ so $r = g'^\sigma = g'$. But clearly the only elements of F fixed by σ are members of k . Thus $g \in (x-1)S + k$ so $T = (x-1)S + k$. We can therefore say: *Any k -algebra $K \cong (x-1)S$ is an a.m.e. algebra for an arbitrary field k , and is an m.e. algebra when $k = Z_p$ for some prime p .*

Note that since $(x-1)S$ contains no idempotents it is distinct from the rings of Example 1. Also note (see [2]) that $(x-1)S$ is a simple algebra (Noetherian when k is a finite field) and hence could be used in Corollaries 4 and 5.

Example 3. [5, page 55.] Let k be any field of characteristic zero and let A be the ring of polynomials in x over $F=k(y)$ where for all $f \in F$ we have $xf = fx + f'$ with f' the formal derivative. Thus $xf = gx$ for $f, g \in F$ implies $f = g$ and $g' = 0$, that is $g \in k$. Since A admits a division algorithm and x is regular in A , an argument similar to that of Example 2 shows that xA is an a.m.e. algebra. Since A is simple, xA is another example of a simple a.m.e. algebra without unit. It is clearly distinct from the algebra of Example 1 and is also not isomorphic to the algebra of Example 2 since there we have the relation $uv = v - u$ where $u = (x-1)x^{-1}$ and $v = x-1$. However, a degree argument shows that such a relation is impossible for non-zero elements of xA .

Example 4. [5, page 113.] To show that there exist non-simple m.e. and a.m.e. algebras over any field k let F be the same field as in Example 2 but now let S be the skew polynomials in x over F with $fx = xf^\sigma$ for all $f \in F$. Since S admits a division algorithm and as before $f = f^\sigma$ if and only if $f \in k$, the argument of Example 2 again shows that $(x-1)S$ is an a.m.e. algebra over k (or an m.e. algebra if $k = \mathbb{Z}_p$). An easy degree argument shows that $(x-1)S$ has proper ideals, such as the ideal generated by $(x-1)x$.

As an application of our minimally embeddable algebras we will answer certain open questions.

In [3] a ring R is called *almost nilpotent* if every proper homomorphic image is nilpotent, and we will call R *almost nil* if every such image is a nil ring. Note that by this definition all simple rings are almost nilpotent.

Application 1. It is well-known that the ring of all linear transformations of an infinite dimensional vector space can be embedded as the heart of a subdirectly irreducible almost nilpotent ring. T. L. JENKINS has asked (see [4]) if this is true for every simple ring without unit. Our minimally embeddable rings (Example 1 or 2) clearly give a negative answer, and also the same answer for a similar question: can every simple ring without unit be the maximal ideal of an almost nilpotent local ring?

Application 2. Let P be an arbitrary (Kuroš—Amitsur) radical class with semi-simple class SP . In [3] were considered radicals with property:

- (L) Every almost nilpotent ring is either in P or in SP , and we will also consider the property:
- (L') Every almost nil ring is either in P or in SP .

It is easy to see that if P contains all zero rings then P has property (L), and Richard Wiegandt has asked [4] if the converse is true at least for P a hereditary radical. That the answer is again negative follows from:

Proposition 7. *Let $L\{K\}$ be the lower radical defined by $\{K\}$ where K is a non-nil simple m.e. algebra over the field k . Then $L\{K\}$ is a hereditary radical containing no zero rings and having properties (L) and (L').*

PROOF. Since K is simple it is well-known that $L\{K\}$ is hereditary and contains no zero rings. Thus suppose R is an almost nil ring with $R \notin SL\{K\}$. Then R has a non-zero radical and hence an accessible subring $I \cong K$. But then I is simple

so $I \triangleleft R$. Now if $I \neq R$ then we would either have a non-zero $B \triangleleft R$ with $I \oplus B = R$ so $R/B \cong I$ would be a non-nil proper image, or else for some $B \triangleleft R$ we would have $R/(I \oplus B) \cong k$ which would also be a non-nil proper image. We conclude that if $R \notin SL\{K\}$ then $R = I \in L\{K\}$.

References

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