

On some properties of group rings

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1. Introduction

The purpose of this paper is twofold. Firstly we show that there are only finitely many conjugacy classes of group bases in the integral group ring ZG of a finite group G . Secondly, as a generalization of the WHITCOMB's result [9] we prove that a finite metabelian group G is determined by the group ring RG where $R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, (b, |G|) = 1 \right\}$. We also apply the proof of this result to show that a group G of order 2^n , $n \leq 7$, is determined by its integral group ring. For certain integral domains K we prove that any finite group which is the group of all units of some K -algebra is determined by the group ring KG .

In what follows KG denotes a group ring of a group G over an associative ring K with 1, $I(K, G)$ stands for the augmentation ideal of KG . The equality $KG = KH$ means that H is a normalised group bases of KG . We shall often write $I(G)$ instead of $I(K, G)$ when a precise situation will be clear from the context. If A is an ideal of KG and S is a subset of KG then put $G \cap 1 + A = \{g \in G \mid g - 1 \in A\}$, $S + A = \{s + A \mid s \in S\}$. The group of all automorphisms of ZG and the group of inner automorphisms of ZG will be denoted by $\text{Aut}(ZG)$ and $\text{In}(ZG)$ respectively. Finally, O_p (respectively $Z_{(p)}$) stands for the ring of p -adic integers (respectively p -integral rationals) and $U(K)$ for the group of units in K .

2. Conjugacy classes of group bases in ZG

Let $ZG = ZH$ and let $G \cong H$. It is natural to ask whether there is a unit u in ZG such that $H = u^{-1}Gu$. That this is not always the case was first proved in 1966 by S. D. BERMAN and A. R. ROSSA ([2] Theorem 4). Therefore we are led to ask whether for an arbitrary finite group G the number of conjugacy classes of group bases in ZG is finite or infinite. The following theorem gives a positive answer to this question.

Theorem 1. *There are only finitely many conjugacy classes of group bases in ZG .*

PROOF. It follows from Theorems I and II of [4] that the group $\text{Aut}(ZG)/\text{In}(ZG)$ is finite. Let $\text{Aut}(ZG) = \text{In}(ZG) + \text{In}(ZG)\varphi_2 + \dots + \text{In}(ZG)\varphi_t$ be the coset decomposition of $\text{Aut}(ZG)$ with respect to $\text{In}(ZG)$. Suppose that H is an arbitrary group basis of ZG . Since $|H| = |G|$ there exists only a finite number of nonisomorphic

group bases in ZG , say, $G_1, G_2, \dots, G_n$. Hence $H \cong G_i$ for some $i \in \{1, 2, \dots, n\}$ and therefore there exists $f \in \text{Aut}(ZG)$ such that $f(G_i) = H$. Since $f = \theta \varphi_j$ for some $\theta \in \text{In}(ZG)$ and some $j \in \{1, 2, \dots, t\}$ then $f(G_i) = u^{-1} \varphi_j(G_i) u$ for some $u \in U(ZG)$, i.e. H is conjugate to $\varphi_j(G_i)$, proving the theorem.

3. The isomorphism problem for the group rings over some integral domains

The isomorphism problem for the group rings asks whether $KG = KH$ implies $G \cong H$. When this is so for a group G , G is said to be determined by the group ring KG . Call the ring K a $(*)$ -ring if for every finite group G the coefficient of 1 in any periodic element x of $U(KG)$ is equal to 0 unless x is of the form $\alpha \cdot 1$ for some $\alpha \in K$.

Lemma 1. *Let N be a normal subgroup of a finite group G and let $\pi: KG \rightarrow K(G/N)$ be a canonical homomorphism where K is a $(*)$ -ring. If $KG = KH$ then the following properties hold:*

- (1) $K(G/N) = K\pi(H)$, $KG \cdot I(N) = KH \cdot I(N^*)$ and $|N| = |N^*|$ where $N^* = H \cap 1 + KG \cdot I(N)$. Moreover, every normal subgroup of H is of the form N^* for some $N \triangleleft G$.
- (2) Periodic elements of the centre of $U(KG)$ are trivial.

By applying the same arguments as in the proof of Lemma 3.1 of [3] we see that $\pi(H)$ is a linearly independent set of the group ring $K(G/N)$. Hence $K(G/N) = K\pi(H)$ and therefore π can be regarded as the extension of the epimorphism $H \rightarrow \pi(H)$ by K -linearity. This shows that $\text{Ker } \pi = KH \cdot I(N^*) = KG \cdot I(N)$. The equality $|N| = |N^*|$ is a consequence of the isomorphism $\pi(H) \cong H/N^*$.

Now if $S \triangleleft H$ then there exists $N \triangleleft G$ such that $KG \cdot I(N) = KH \cdot I(S)$. Hence $KH \cdot I(S) = KH \cdot I(N^*)$ and therefore $S = H \cap 1 + KH \cdot I(S) = H \cap 1 + KH \cdot I(N^*) = N^*$, proving (1). The proof of (2) is evident.

Let K be a commutative ring. We call a group G a unit group over K if G is isomorphic to the group of all units of some K -algebra. By taking the case $K = \mathbb{Z}$ in the following theorem we obtain another proof of the characterization theorem for the unit groups due to R. SANDLING [7].

Theorem 2. *Let G be a finite group and let A be a K -algebra, where K is an integral domain of characteristic 0 in which no prime dividing the order of G is invertible. If $KG = KH$ and if $\mu: G \rightarrow U(A)$ is a monomorphism then the mapping $\mu': H \rightarrow U(A)$, given by $\mu(\sum_g \alpha_g g) = \sum_g \alpha_g \mu(g)$ for any $h = \sum_g \alpha_g g \in H$ is also a monomorphism.*

In particular, a finite group G which is a unit group over K is determined by the group ring KG .

PROOF. Let $\mu^*: KG \rightarrow A$ be the homomorphism of K -algebras obtained from μ by extension by K -linearity. Then μ' is the restriction of μ^* to H and therefore μ' is a homomorphism. It follows from [6] that K is a $(*)$ -ring. Therefore by Lemma 1 we have $\text{Ker } \mu' = N_1^*$ and $KG \cdot I(N_1) = KH \cdot I(N_1^*)$ for some $N_1 \triangleleft G$. Since $I(N_1^*) \subseteq \text{Ker } \mu^*$ then $I(N_1) \subseteq \text{Ker } \mu^*$ and therefore $N_1 \subseteq \text{Ker } \mu = 1$. Thus $N_1^* = 1$, proving the theorem.

Corollary. Let $\bar{B}$ be a subgroup of a finite abelian group B and let G be isomorphic to the group L of all automorphisms of B which leave $\bar{B}$ invariant (as a set). Then G is determined by its integral group ring.

PROOF. Let $A = \text{Hom}(B, B)$. Then the isomorphism $G \cong L$ induces monomorphism $G \rightarrow U(A)$. By Theorem 2 (with $K = \mathbb{Z}$), the equality $ZG = ZH$ implies $\mu(G) = \mu'(H)$ and therefore $G \cong H$.

For the proof of our next theorem we need the following lemmas.

Lemma 2. Let G be an arbitrary group, K an arbitrary ring with 1, N arbitrary subgroup of G . Then in the group ring KG the following equalities hold:

$$(3) \quad I(G) \cdot I(N) \cap I(N) = I(N)^2$$

$$(4) \quad G \cap 1 + I(G) \cdot I(N) = N \cap 1 + I(N)^2.$$

PROOF. Since $G \cap 1 + KG \cdot I(N) = N$ it follows that $G \cap 1 + I(G) \cdot I(N) = N \cap 1 + I(G) \cdot I(N) = N \cap 1 + (I(G) \cdot I(N) \cap I(N))$ and therefore (3) $\Rightarrow$ (4). Let T be a full set of cosets representatives of G with respect to N .

If $g = tn$ where $n \in N$, $t \in T$ then for $n' \in N$ we have $(g-1)(n'-1) = (t-1)(n-1)(n'-1) + (t-1)(n'-1) + (n-1)(n'-1)$. Since the first and the second summands belong to $(t-1) \cdot I(N)$ and since $(n-1)(n'-1) \in I(N)^2$ then

$$(g-1)(n'-1) \in I(N)^2 + (t-1)I(N)$$

from which follows that

$$I(G) \cdot I(N) = I(N)^2 + \sum_{1 \neq t \in T} (t-1)I(N).$$

Let

$$x = y + (t_1-1)[\alpha_{11}(n_1-1) + \dots + \alpha_{1s}(n_s-1)] + \dots + (t_k-1)[\alpha_{s1}(n_1-1) + \dots + \alpha_{ss}(n_s-1)]$$

where $y \in I(N)^2$, $t_j \in T$, $n_i \in N$, $1 \leq i \leq s$, $1 \leq j \leq k$. If $x \in I(N)$ then $z = \alpha_{11}t_1(n_1-1) + \dots + \alpha_{1s}t_s(n_s-1) + \dots + \alpha_{ss}t_k(n_s-1) \in I(N)$ and since all elements of N have coefficient 0 in z , then $z = 0$. But $\{t_1(n_1-1), \dots, t_s(n_s-1)\}$ is a linearly independent set and therefore $\alpha_{11} = \dots = \alpha_{1s} = \dots = \alpha_{ss} = 0$. Hence $x = y \in I(N)^2$, proving the lemma.

Lemma 3. Let G be a group containing an abelian subgroup A of a finite exponent n and let $K = \mathbb{Z}/m\mathbb{Z}$ where $m \equiv 0 \pmod{n}$. Then the following properties hold:

$$(5) \quad G \cap 1 + I(G) \cdot I(A) = 1.$$

(6) If $x \in KG$ and if $x \equiv g \pmod{KG \cdot I(A)}$ for some $g \in G$ then there exists an element $g_x = ga$ ($a \in A$) such that $x \equiv g_x \pmod{I(G) \cdot I(A)}$.

PROOF. As in the case $K = \mathbb{Z}$, the formula

$$f\left(\sum_{a \in A} (\alpha_a \cdot 1)(a-1)\right) = \prod_{a \in A} a^{\alpha_a} \quad (\alpha_a \in \mathbb{Z})$$

defines a homomorphism of $I(A)$ onto A with kernel $I(A)^2$. From this follows that $A \cap 1 + I(A)^2 = 1$ and the application of (3) yields (5).

Since $f(\sum_{a \in A} (\alpha_a \cdot 1)(a-1)) = f(\prod_{a \in A} a^{\alpha_a} - 1)$ then

$$(7) \quad \sum_{a \in A} (\alpha_a \cdot 1)(a-1) \equiv \prod_{a \in A} a^{\alpha_a} - 1 \pmod{I(A)^2} (\alpha_a \in \mathbb{Z}).$$

Note also that $KG = K + I(G)$ whence $KG \cdot I(A) = I(A) + I(G) \cdot I(A)$ and $x \equiv g + t \pmod{I(G) \cdot I(A)}$ for some $t = \sum_{s \in A} (\alpha_s \cdot 1)(s-1) \in I(A)$. Applying (7) we obtain

$$x \equiv g + (a-1) = (1-g)(a-1) + ga \equiv g_x \pmod{I(G) \cdot I(A)}$$

where $a = \prod_{s \in A} s^{\alpha_s}$. This completes the proof of the lemma.

Let K be a commutative ring and let $\varphi: K \rightarrow \bar{K}$ be the ring epimorphism. If $x = \sum_g \alpha_g g \in KG$ then put $\bar{x} = \sum_g \bar{\alpha}_g g$ where $\bar{\alpha}_g = \varphi(\alpha_g)$.

It is clear that the mapping $\lambda: KG \rightarrow \bar{K}G$, defined by $\lambda(x) = \bar{x}$ for any $x \in KG$ is a ring epimorphism. We call λ the projection of KG onto $\bar{K}G$. Suppose that Λ is an ideal of the group algebra $\bar{K}G$. Then the ring $\bar{K}G/\Lambda$ can be regarded as a K -algebra in the obvious way. Moreover, the mapping $\varphi^*: KG \rightarrow \bar{K}G/\Lambda$ defined by $\varphi^*(x) = \bar{x} + \Lambda$ is a K -algebra homomorphism. We are now ready to prove the following result.

Theorem 3. *A finite metabelian group G is determined by the group ring RG where $R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, (b, |b|) = 1 \right\}$.*

PROOF. Let $RG = RH$. The mapping $\varphi: R \rightarrow \bar{R} = \mathbb{Z}/|G|\mathbb{Z}$ defined by $\varphi\left(\frac{a}{b}\right) = \bar{a}(\bar{b})^{-1}$ where $\bar{a} = a + |G|\mathbb{Z}$ is a ring epimorphism. Consider the mapping φ^* defined as above by taking $\Lambda = I(\bar{R}, G) \cdot I(\bar{R}, G')$. It follows from (5) and the Theorem 2 that the restrictions of φ^* to G and H induce group isomorphisms $G \rightarrow \varphi^*(G)$ and $H \rightarrow \varphi^*(H)$, where $\varphi^*(G) = G + \Lambda$, $\varphi^*(H) = \bar{H} + \Lambda$, $\bar{H} = \{\bar{h} \mid h \in H\}$. Since R is a $(*)$ -ring [6] the application of (2) to $R(G/G')$ yields $G + RG \cdot I(R, G') = H + RG \cdot I(R, G')$.

Therefore projecting RG onto $\bar{R}G$ we obtain $G + \bar{R}G \cdot I(\bar{R}, G') = \bar{H} + \bar{R}GI(\bar{R}, G')$. It follows from (6) that in this case $\bar{H} + \Lambda \subseteq G + \Lambda$, i.e. $\varphi^*(H) \subseteq \varphi^*(G)$. But $|G| = |\bar{H}|$ and therefore $G \cong H$, proving the theorem.

In [8] W. R. WELLER proved that there are only two nonconjugate classes of normalised group bases of ZD_4 , where D_4 is a dihedral group of order 8. By combining this result with the Theorem 4 of [2] we obtain the following property:

$$(8) \quad \text{Every two normalised group bases of } ZD_4 \text{ are conjugate in } U(Z_{(2)}D_4).$$

Theorem 4. *Let $|G| = 2^n$, $n \leq 7$. Then the group G is determined by its integral group ring.*

PROOF. Every group of order 2^n , $n \leq 6$ is metabelian and a group G of order 2^7 has a normal abelian subgroup A of index 8 ([5], p. 120). Hence we can restrict ourselves to the case when $|G| = 2^7$ and the factor group G/A is nonabelian of

order 8. Let $ZG = ZH$. It follows from the proof of Theorem 3 that

$$(9) \quad G \cong H \text{ whenever } G + RG \cdot I(R, A) = H + RG \cdot I(R, A).$$

If G/A is the quaternion group then by [1] $G + ZG \cdot I(Z, A) = H + ZG \cdot I(Z, A)$. Thus we have only to consider the case when $G/A \cong D_4$. Let $\pi: ZG \rightarrow Z(G/A)$ be the canonical homomorphism. It follows from (1) that $Z\pi(G) = Z\pi(H)$ and therefore $Z_{(2)}\pi(G) = Z_{(2)}\pi(H)$.

By (8) there exists a unit $u \in Z_{(2)}\pi(G)$ such that $u^{-1}\pi(H)u = \pi(G)$ and since $Z_{(2)}D_4$ is a local ring then $u = \pi(t)$ for some $t \in Z_{(2)}G$. Therefore $\pi(t^{-1}Ht) = \pi(G)$ and $t^{-1}Ht + RG \cdot I(R, A) = G + RG \cdot I(R, A)$ for $R = Z_{(2)}$. It follows from (9) that in this case $G \cong t^{-1}Ht$, proving the theorem.

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