

On some classification of the linear connection in the small-dimensional space L_n

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§ 1. Introduction

Let be given at point $\overset{0}{\xi}$ of differentiable manifold X_n an abstract geometric object, special, purely differential in with the space of coordinates (fibre) $\mathfrak{M}$ and transformation rule

$$(1.1) \quad \omega' = F(\omega, L), \quad \omega \in \mathfrak{M}, \quad L \in L_n^s,$$

where L_n^s is n -dimensional differential group of rank s , [1].

The set of all elements of space $\mathfrak{M}$ obtaining for fixed $\omega_0 \in \mathfrak{M}$ by action of all elements $L \in L_n^s$:

$$(1.2) \quad \omega = F(\omega_0, L), \quad L \in L_n^s, \quad \omega_0 \in \mathfrak{M}$$

is called an orbit of space $\mathfrak{M}$ and is denoted by symbol $\mathfrak{M}_{\omega_0}$, [1].

Obviously we have distribution

$$(1.3) \quad \mathfrak{M} = \bigcup_{\omega_0 \in \mathfrak{M}} \mathfrak{M}_{\omega_0}.$$

The subset $\overline{\mathfrak{M}} \subset \mathfrak{M}$ is called an invariant set of an abstract object ω if there is satisfied implication:

$$(1.4) \quad \text{if } \omega_1 \in \overline{\mathfrak{M}}, \text{ then } \omega_2 = F(\omega_1, L) \in \overline{\mathfrak{M}} \text{ for every } L \in L_n^s.$$

The invariant set $\overline{\mathfrak{M}}$ is also the sum of some number of orbits.

The space with linear connection L_n is the differential manifold X_n provided with a field of linear connection L_{ij}^k satisfying the following transformation rule:

$$(1.5) \quad L_{i'j'}^{k'} = A_{i'}^i A_{j'}^j A_k^{k'} L_{ij}^k + A_l^{k'} \partial_{i'} A_{j'}^l, \quad \begin{array}{l} i, j, k, l = 1, \dots, n \\ i', j', k' = 1', \dots, n', \end{array}$$

where

$$(1.6) \quad A_k^{k'} = \frac{\partial \varphi^{k'}(\xi^k)}{\partial \xi^k}, \quad A_{i'}^i = \frac{\partial \psi^i(\xi^{i'})}{\partial \xi^{i'}}$$

are partial derivatives of transformation of coordinate system $(\kappa) \rightarrow (\kappa')$ determined in some neighbourhood of every point $\xi \in X_n$ [3].

Moreover the formula is true

$$(1.7) \quad A_{i'}^i = \frac{\text{minor } A_i^{i'}}{J}, \quad J = \det(A_i^{i'}) \neq 0,$$

where minor is understood in algebraical sense.

We take denotations

$$(1.8) \quad L_{ij}^k = \Gamma_{ij}^k + S_{ij}^k,$$

where

$$(1.9) \quad \Gamma_{ij}^k = L_{(ij)}^k, \quad S_{ij}^k = L_{[ij]}^k.$$

On base of [3] it is known that Γ_{ij}^k is linear symmetric connection and torsion S_{ij}^k is antisymmetric tensor field.

In view of papers [4], [5] at every fixed point $\xi^0 \in X_n$ are known the canonical forms and the orbits (the families of orbits) of the coordinate space $\mathfrak{M}$ of anti-symmetric tensor S_{ij}^k for $n=2$ and $n=3$. Basing on this we shall carry out the classification of linear connection L_{ij}^k in next sections of the paper.

Definition 1. Linear connection L_{ij}^k of space L_n is said to be of type M_S^0 corresponding with canonical form S_0^0 of the torsion tensor S_{ij}^k , where $\mathfrak{M}_S^0$ is an orbit (a family of orbit) of the coordinates space of torsion S_{ij}^k generated by S_0^0 , if at every point $\xi^0 \in L_n$, $[S_{ij}^k] \in \mathfrak{M}_S^0$.

Basing on definition 1 we can say that space L_n is of type $\mathfrak{M}_S^0$.

§ 2. Classification of space L_2

In space L_2 tensor S_{ij}^k of type (1.9) has only two essential coordinates S_{12}^1 , S_{12}^2 and the matrix of its coordinates is of the form

$$(2.1) \quad S = \begin{bmatrix} 0 & 0 \\ S_{12}^1 & S_{12}^2 \\ -S_{12}^1 & -S_{12}^2 \\ 0 & 0 \end{bmatrix}.$$

Taking denotations

$$(2.2) \quad S^k = \frac{1}{2} S_{ij}^k \varepsilon^{ij} \quad k, i, j = 1, 2$$

we obtain the relation

$$(2.3) \quad S_{ij}^k = S^k \varepsilon_{ij},$$

where in formulas (2.2) and (2.3) ε^{ij} and ε_{ij} are Ricci symbols of space X_2 .

On base of [2], [3] the abstract object (2.2) has the following transformation rule

$$(2.4) \quad S^{k'} = J^{-1} A_k^{k'} S^k, \quad k = 1, 2; \quad k' = 1', 2'.$$

Object (1.9) is vector- G -density of weight (1). In virtue of (2.2) and (2.3) we can see that abstract objects S_{ij}^k and S^k are strictly equivalent, therefore they have the same number of homeomorphic in pairs orbits, [1].

Object S^i has two orbits:

$$(2.5) \quad \mathfrak{N}_1 = (0, 0), \quad \mathfrak{N}_2 = R^2 - \{(0, 0)\}.$$

The represent of orbit $\mathfrak{N}_2$ is element $(0, 1)$ determining the second canonical form of object S^i .

Really, for $(S^1, S^2) \neq (0, 0)$ the following nonsingular matrix [5]

$$(2.6) \quad [A_i^{i'}] = \begin{bmatrix} S_{12}^2 & -S_{12}^1 \\ S_{12}^1 & S_{12}^2 \end{bmatrix}$$

transforms with the help of rule (2.4) an arbitrary point $S_1, S_2 \neq (0, 0)$ into point $(0, 1)$.

In view of the above tensor S_{ij}^k has the following canonical forms:

$$(2.7) \quad S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

In view (2.1) and (2.6) tensor S_{ij}^k has two orbits determined below by conditions:

$$(2.8) \quad \mathfrak{M}_{S_1}: (S^1)^2 + (S^2)^2 = 0;$$

$$(2.9) \quad \mathfrak{M}_{S_2}: (S^1)^2 + (S^2)^2 > 0.$$

Directly from definition 1 (§ 1) and from formulas (2.8) and (2.9) results the following

Theorem 1. Space L_2 is of the type $\mathfrak{M}_1$ or $\mathfrak{M}_2$ when at every point $\xi \in L_2$ matrix S of the form (2.1) satisfies the condition: $S \in \mathfrak{M}_1$ or $S \in \mathfrak{M}_2$.

§ 3. Orbits of tensor S_{ij}^k in space X_3 .

In space X_3 tensor S_{ij}^k of the type (1.9) has nine essential coordinates. Introducing denotations

$$(3.1) \quad S^{ij} = \frac{1}{2} S_{pq}^i \varepsilon^{pqj}$$

we get the relations

$$(3.2) \quad S_{ij}^k = S^{kl} \varepsilon_{ijl}, \quad i, j, k, l, p, q = 1, 2, 3$$

where $\varepsilon^{ijk}, \varepsilon_{ijk}$ are Ricci symbols of space X_3 [2].

Objects S_{ij}^k and S_{ij} are strictly equivalent.

On base of [1] the abstract object (1.3) has the transformation rule

$$(3.3) \quad S^{i'j'} = J^{-1} A_i^{i'} A_j^{j'} S^{ij}.$$

In paper [4] is proved the following

Theorem 2. *In space X_3 tensor $-G$ -density (3.3) has eighteen following canonical forms (congruent) which we will write in an integrate form:*

$$(3.4) \quad \left\{ \begin{array}{ll} S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, & S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \\ \\ \text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; & \text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; \\ \\ S_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, & S_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \\ \\ S_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, & S_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1+\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \quad \alpha \neq 0, \\ \\ S_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & -\beta & 1 \end{bmatrix}, & S_{10} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta \neq 0, \\ \\ S_{11} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, & S_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1+\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \\ \\ \text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; & \alpha \neq 0 \\ \\ S_{13} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & \beta \\ 0 & -\beta & 1 \end{bmatrix}, & S_{14} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \\ \\ \text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; \end{array} \right.$$

Theorem 3. At every point $\overset{0}{\xi} \in X_3$ the orbits $\mathfrak{M}_{S_i}$, $i=1, \dots, 5$ of the tensor-density (3.1) can be characterized in the following way:

$$(3.5) \quad \left\{ \begin{array}{l} \mathfrak{M}_{S_1}: p_1 = p_2 = p_3 = 0; \\ \mathfrak{M}_{S_2}: p_1 \neq 0, \quad p_2 = p_3 = 0; \\ \mathfrak{M}_{S_3}^a: p_1 \neq 0, \quad p_2 > 0, \quad p_3 = 0; \\ \mathfrak{M}_{S_3}^b: p_1 \text{-arbitrary}, \quad p_2 < 0, \quad p_3 = 0; \\ \mathfrak{M}_{S_4}^a: p_1 \neq 0, \quad p_2 < 0, \quad p_3 \neq 0; \\ \mathfrak{M}_{S_4}^b: \begin{cases} p_1 \neq 0, \quad p_2 \equiv 0, \quad p_3 \neq 0; \\ \text{or} \\ p_1 \text{-arbitrary}, \quad p_2 < 0, \quad p_3 \neq 0; \end{cases} \\ \mathfrak{M}_{S_5}: S^{(ij)} = 0, \quad S^{[ij]} \neq 0, \end{array} \right.$$

where $-p_1, p_2, -p_3$ are the coefficients of the characteristic polynomial of matrix $S=[S^{ij}]$:

$$(3.6) \quad \varphi(\lambda) = \det(\lambda E - S) = \lambda^3 - p_1 \lambda^2 + p_2 \lambda - p_3.$$

To characterize the other orbits (families of orbits) of tensor $-G$ -density (3.1) we will define at first some auxiliary geometric objects. Let

$$(3.7) \quad h^{ij} = S^{(ij)}, \quad k^{ij} = S^{[ij]}, \quad i, j = 1, 2, 3.$$

On base of (3.1) and (3.3) we observe that objects h^{ij} and k^{ij} are also tensor $-G$ -densities of weight (1).

Let us the tensor bundle $\lambda h^{ij} + k^{ij}$ or corresponding with it the matrix bundle $\lambda H + K$, where

$$(3.8) \quad H = [h^{ij}], \quad K = [k^{ij}].$$

For matrix bundle $\lambda H + K$ let us form a compound matrix of degree 2, ..., where $S_{ij}(\lambda)$ is minor of degree 2 of matrix $\lambda H + K$ arised by drawing out of i -th row and j -th column; $i, j = 1, 2, 3$.

On base of immediately calculations we obtain the following distribution:

$$(3.9) \quad S_{ij}(\lambda) = \lambda^2 H_{ij} + \lambda M_{ij} + K_{ij}, \quad i, j = 1, 2, 3$$

or in matrix form

$$(3.10) \quad S(\lambda) = \lambda^2 H + \lambda M + K,$$

where

$$(3.11) \quad H = [H_{ij}], \quad K = [K_{ij}], \quad M = [M_{ij}],$$

while in view of (3.7) and (3.10) for $\lambda = 1$

$$(3.12) \quad M = S - H - K.$$

Matrices H, K and M satisfy the following conditions of symmetry or anti-symmetry:

$$(3.13) \quad H^t = H, \quad K^t = K, \quad M^t = -M.$$

In paper [4] is proved that geometric objects $S_{ij}(\lambda), H_{ij}, M_{ij}, K_{ij}$ defined by distribution (3.9) have the transformation rules of following type:

$$(3.14) \quad M_{i'j'} = B_{i'}^i B_{j'}^j M_{ij}, \quad i, j = 1, 2, 3; \quad i', j' = 1', 2', 3'$$

where

$$(3.15) \quad B_{i'}^i = (-1)^{i+i'} A_{i'}^i, [A_{i'}^i] \in GL(3, R).$$

They are the geometric linear homogeneous objects therefore their vanishing or non-vanishing has invariant character.

Now we can formulate the following

Theorem 4. *At every point $\xi \in X_3$ the orbits (families of orbits) $\mathfrak{M}_{S_l}, l=6, \dots, 14$ of tensor-G-density (3.1) can be characterized by the following systems of conditions:*

$$(3.16) \quad \left\{ \begin{array}{l} \mathfrak{M}_{S_6}: H \in \mathfrak{M}_{S_3}^b, \quad K \neq 0, \quad M \neq 0, \quad \det S = 0; \\ \mathfrak{M}_{S_7}: H \in \mathfrak{M}_{S_2}, \quad K \neq 0, \quad M = 0; \\ \mathfrak{M}_{S_8}: H \in \mathfrak{M}_{S_3}^b, \quad K \neq 0, \quad M = 0, \quad K = -\alpha^2 H, \quad \alpha \neq 0; \\ \mathfrak{M}_{S_9}: H \in \mathfrak{M}_{S_3}^a, \quad K \neq 0, \quad M = 0, \quad K = \beta^2 H, \quad \beta \neq 0; \\ \mathfrak{M}_{S_{10}}: H \in \mathfrak{M}_{S_2}, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}_{S_{11}}^a: H \in \mathfrak{M}_{S_3}^a, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}_{S_{11}}^b: H \in \mathfrak{M}_{S_3}^b, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}_{S_{12}}: H \in \mathfrak{M}_{S_4}^b, \quad \det S \neq \det H, \quad \alpha^2 = \frac{\det H - \det S}{\det H}; \\ \mathfrak{M}_{S_{13}}^a: H \in \mathfrak{M}_{S_4}^a, \quad \det S \neq \det H, \quad \beta^2 = \frac{\det S - \det H}{\det H}; \\ \mathfrak{M}_{S_{13}}^b: H \in \mathfrak{M}_{S_4}^b, \quad \det S \neq \det H, \quad \beta^2 = \frac{\det S - \det H}{\det H}; \\ \mathfrak{M}_{S_{14}}: H \in \mathfrak{M}_{S_4}^b, \quad K \neq 0, \quad \det S = \det H. \end{array} \right.$$

§ 4. Classification of space L_3

In view of the results of section 3 we can make the classification of the linear connection L_{ij}^k of space L_3 .

In virtue of formulas (3.1) or (3.2) we can see that tensor S_{ij}^k and G-density S^{ij} are the abstract geometric objects strictly equivalent [1]. In that case both

geometric objects S_{ij}^k and S^{ij} have the same homeomorphic in pairs orbits (families of orbits) what permits to do the mentioned below classification.

According to formula (3.2) matrix $\hat{S}$ of coordinates of tensor S_{ij}^k can be expressed with the help of coordinates of tensor- G -density S^{ij} in the following way:

$$(4.1) \quad \hat{S} = \begin{bmatrix} 0 & 0 & 0 & -S^{13} & -S^{23} & -S^{33} & S^{12} & S^{22} & S^{32} \\ S^{13} & S^{23} & S^{33} & 0 & 0 & 0 & -S^{11} & -S^{21} & -S^{31} \\ -S^{12} & -S^{22} & -S^{32} & S^{11} & S^{21} & S^{31} & 0 & 0 & 0 \end{bmatrix}.$$

Definition 2. The canonical form of tensor S_{ij}^k we call the form $\hat{S}_0$ of matrix (4.1) corresponding with canonical form of coordinate matrix $[S^{ij}]$ [of types (3.5)] of tensor- G -density S^{ij} determined by formulas (3.1) and (3.3).

On base of formulas (3.4) and (4.1) we have the following

Corollary. Antisymmetric tensor S_{ij}^k has at every point $\xi \in X_3$ eighteen canonical forms $\hat{S}_0$ of type [(4.1), (3.4)].

In accordance with definition 1 from section 1 and with theorems 3 and 4 from section 3 is true the following

Theorem 5. In space L_3 we can distinguish eighteen types of linear connection $L_{ij}^k(\xi^l)$ in dependency of the fact that matrix $\hat{S}$ of the tensor of torsion S_{ij}^k belongs at each point $\xi \in L_3$ to the some orbit (family of orbits) $\mathfrak{M}_{s_1}, \dots, \mathfrak{M}_{s_{14}}$ defined by formulas (3.5) or (3.16) respectively.

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