

Some general functional equations

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Introduction

The purpose of this paper is to find the general solution of the functional equations

$$(1) \quad \begin{cases} \prod_{i=1}^n f_i(x_i) = g\left(\sum_{i=1}^n x_i\right) G(x_1 - x_n, \dots, x_{n-1} - x_n); & x_1, \dots, x_n \in \mathbf{R}, \\ f_i: \mathbf{R} \rightarrow \mathbf{R} \quad (i = 1, \dots, n), & g: \mathbf{R} \rightarrow \mathbf{R}, \quad G: \mathbf{R}^{n-1} \rightarrow \mathbf{R} \end{cases}$$

and

$$(2) \quad \begin{cases} \prod_{i=1}^n f_i(x_i) = g\left(\sum_{i=1}^n x_i\right) G\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right); & x_1, \dots, x_n \in \mathbf{R}_+, \\ f_i: \mathbf{R}_+ \rightarrow \mathbf{R} \quad (i = 1, \dots, n), & g: \mathbf{R}_+ \rightarrow \mathbf{R}, \quad G: \mathbf{R}_+^{n-1} \rightarrow \mathbf{R}, \end{cases}$$

where $\mathbf{R}$ is the set of real numbers and $\mathbf{R}_+ = \{x | x > 0, x \in \mathbf{R}\}$.

Equations (1) and (2) have applications to the characterization of normal and gamma distributions (see [5]).

1. Preliminary results

We need the following results:

Theorem 1.1. (see [7]). Suppose that the functions $f, g, h, k: \mathbf{R} \rightarrow \mathbf{R}$ satisfy the functional equation

$$(1.1) \quad f(x)g(y) = h(ax + by)k(cx + dy) \quad (x, y \in \mathbf{R}),$$

where $a, b, c, d \in \mathbf{R}_0 = \mathbf{R} \setminus \{0\}$ are arbitrary rational constants with $\Delta = ad - bc \neq 0$ and there exists a subset $D \subset \mathbf{R}^2$ of positive Lebesgue-measure, such that $f(x)g(y) \neq 0$ for all $(x, y) \in D$, then

$$(1.2) \quad f(x) = \alpha_1 \exp[A_1(x) + n(x)] \quad (x \in \mathbf{R}),$$

$$(1.3) \quad g(x) = \alpha_2 \exp\left[A_2(x) - \frac{bd}{ac} n(x)\right] \quad (x \in \mathbf{R}),$$

$$(1.4) \quad h(x) = \beta_1 f\left(\frac{d}{\Delta} x\right) g\left(-\frac{c}{\Delta} x\right) \quad (x \in \mathbf{R}),$$

$$(1.5) \quad k(x) = \beta_2 f\left(-\frac{b}{\Delta} x\right) g\left(\frac{a}{\Delta} x\right) \quad (x \in \mathbf{R}),$$

where

$$(1.6) \quad \alpha_1 \beta_1 \alpha_2 \beta_2 = 1$$

holds and the functions $A_i: \mathbf{R} \rightarrow \mathbf{R}$ ($i=1, 2$) and $n: \mathbf{R} \rightarrow \mathbf{R}$ satisfy the functional equations

$$(1.7) \quad A(x+y) = A(x) + A(y) \quad (x, y \in \mathbf{R})$$

and

$$(1.8) \quad n(x+y) + n(x-y) = 2n(x) + 2n(y) \quad (x, y \in \mathbf{R})$$

respectively.

Corollary 1.1 (see [1], [7]). *The general measurable solutions of (1.1) are*

$$(1.9) \quad f(x) = \alpha_1 \exp [a_1 x + b_1 x^2] \quad (x \in \mathbf{R}),$$

$$(1.10) \quad g(x) = \alpha_2 \exp \left[a_2 x - \frac{bd}{ac} b_1 x^2 \right] \quad (x \in \mathbf{R}),$$

$$(1.11) \quad h(x) = \beta_1 \alpha_1 \alpha_2 \exp \left[\frac{a_1 d - a_2 c}{\Delta} x + \frac{d}{a} b_1 x^2 \right] \quad (x \in \mathbf{R}),$$

$$(1.12) \quad k(x) = \beta_2 \alpha_1 \alpha_2 \exp \left[\frac{a_2 a - a_1 b}{\Delta} x - \frac{b}{c} b_1 x^2 \right] \quad (x \in \mathbf{R}),$$

where $a_i, b_i, \alpha_i, \beta_i \in \mathbf{R}$ ($i=1, 2$) are arbitrary constants satisfying (1.6). (Again the existence of $D \subset \mathbf{R}^2$ with positive measure such that $f(x)g(y) \neq 0$ on D is supposed.)

Theorem 1.2 (see [2] and [6]). *Suppose that the functions $f, g, p, q: \mathbf{R}_+ \rightarrow \mathbf{R}$ satisfy the functional equation*

$$(1.13) \quad f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right) \quad (x, y \in \mathbf{R}_+)$$

and there exist sets $T_1, T_2 \subset \mathbf{R}_+$ of positive Lebesgue-measure such that $f(x)g(y) \neq 0$ for all $x \in T_1, y \in T_2$. Then

$$(1.14) \quad f(x) = A \exp [a(x) + m_1(x)] \quad (x \in \mathbf{R}_+),$$

$$(1.15) \quad g(x) = B \exp [a(x) + m_2(x)] \quad (x \in \mathbf{R}_+),$$

$$(1.16) \quad p(x) = C \exp [a(x) + m_1(x) + m_2(x)] \quad (x \in \mathbf{R}_+),$$

$$(1.17) \quad q(x) = D \exp \left[m_1\left(\frac{x}{x+1}\right) - m_2(x+1) \right] \quad (x \in \mathbf{R}_+),$$

where

$$(1.18) \quad AB = CD$$

holds and the functions $a: \mathbf{R} \rightarrow \mathbf{R}$, $m_i: \mathbf{R}_+ \rightarrow \mathbf{R}$ ($i=1, 2$) satisfy the functional equations (1.7) and

$$(1.19) \quad m(xy) = m(x) + m(y) \quad (x, y \in \mathbf{R}_+)$$

respectively.

We remark that a simple proof of Theorem 1.2 can be based on a theorem of Z. DARÓCZY—K. LAJKÓ—L. SZÉKELYHIDI (see [3]).

Corollary 1.2 (see [2]). *The general measurable solutions of (1.13) are the functions*

$$(1.20) \quad f(x) = A \exp [ax + b \ln x] \quad (x \in \mathbf{R}_+),$$

$$(1.21) \quad g(x) = B \exp [ax + c \ln x] \quad (x \in \mathbf{R}_+),$$

$$(1.22) \quad p(x) = C \exp [ax + (c+b) \ln x] \quad (x \in \mathbf{R}_+),$$

$$(1.23) \quad q(x) = D \exp \left[b \ln \frac{x}{x+1} - c \ln (x+1) \right] \quad (x \in \mathbf{R}_+),$$

where $a, b, c \in \mathbf{R}$ and $A, B, C, D \in \mathbf{R}_0$ are arbitrary constants with (1.18). (Again the existence of $T_1, T_2 \subset \mathbf{R}_+$ with positive measure such that $f(x)g(y) \neq 0$ for all $x \in T_1, y \in T_2$ is supposed.)

2. On the functional equation (1)

Lemma 2.1 (see [5]). *Suppose that the functions f_i, g, G satisfy the functional equation (1) and there exist sets $A_i \subset \mathbf{R}$ ($i=1, \dots, n$) of positive Lebesgue-measure such that $\prod_{i=1}^n f_i(x_i) \neq 0$ ($x_i \in A_i; i=1, \dots, n$). Then $\prod_{i=1}^n f_i(x_i)g(u_1)G(u_2, \dots, u_n) \neq 0$ for all $x_i, u_i \in \mathbf{R}$ ($i=1, \dots, n$).*

Using this lemma and the Theorem 1.1, we get the general solution of (1):

Theorem 2.1. *Suppose that the functions f_i, g and G satisfy the conditions of Lemma 2.1. Then we have*

$$(2.1) \quad f_i(x_i) = \alpha_i \exp [a_i(x_i) + n(x_i)] \quad (x_i \in \mathbf{R}, i=1, \dots, n),$$

$$(2.2) \quad g(x) = \beta \prod_{i=1}^n \alpha_i \exp \left[\frac{1}{n} \left(\sum_{i=1}^n a_i(x) + n(x) \right) \right] \quad (x \in \mathbf{R}),$$

$$(2.3) \quad G(x_1, \dots, x_{n-1}) = \alpha_n \frac{\prod_{i=1}^{n-1} f_i(x_i)}{g\left(\sum_{i=1}^{n-1} x_i\right)} \quad (x_i \in \mathbf{R}, i=1, \dots, n-1),$$

where the functions $a_i: \mathbf{R} \rightarrow \mathbf{R}$ ($i=1, \dots, n$) and $n: \mathbf{R} \rightarrow \mathbf{R}$ fulfil the functional equations (1.7) and (1.8) respectively, $\alpha_i, \beta \in \mathbf{R}_0$ ($i=1, \dots, n$) are arbitrary constants.

PROOF. Using Lemma 2.1, it follows that

$$\prod_{i=1}^n f_i(x_i) g(u_1) G(u_2, \dots, u_n) \neq 0$$

for all $x_i, u_i \in \mathbf{R}$ ($i=1, \dots, n$).

Now let $j < n$ be fixed and $x_i = x_n$ if $i \neq j$. From (1) we get

$$f_j(x_j) \prod_{\substack{i=1 \\ i \neq j}}^n f_i(x_n) = g(x_j + (n-1)x_n) G(0, \dots, 0, x_j - x_n, 0, \dots, 0)$$

for all $x_j, x_n \in \mathbf{R}$. This implies the functional equation

$$(2.4) \quad f_j(x_j) \bar{f}_n(x_n) = g(x_j + (n-1)x_n) \bar{G}(x_j - x_n) \quad (x_j, x_n \in \mathbf{R}),$$

where

$$(2.5) \quad \bar{f}_n(x_n) = \prod_{\substack{i=1 \\ i \neq j}}^n f_i(x_n) \quad (x_n \in \mathbf{R}),$$

$$(2.6) \quad \bar{G}(x) = G(0, \dots, 0, \frac{x}{j}, 0, \dots, 0) \quad (x \in \mathbf{R}).$$

The functional equation (2.4) is a special case of (1.1) with $a=1, b=n-1, c=1, d=-1$. Further $\bar{f}_n(x_n) \neq 0$ for all $x_n \in \mathbf{R}$. Thus the functions f_j satisfy the conditions of Theorem 1.1. Therefore $f_j, \bar{f}_n, g$ and $\bar{G}$ are of the forms

$$(2.7) \quad f_j(x_j) = \alpha_j \exp[a_j(x_j) + n_j(x_j)] \quad (x_j \in \mathbf{R}),$$

$$(2.8) \quad \bar{f}_n(x_n) = \alpha_n^j \exp[a_n^j(x_n) + (n-1)n_j(x_n)] \quad (x_n \in \mathbf{R}),$$

$$(2.9) \quad g(x) = \beta_j^1 f_j\left(\frac{x}{n}\right) \bar{f}_n\left(\frac{x}{n}\right) \quad (x \in \mathbf{R}),$$

$$(2.10) \quad \bar{G}(x) = \beta_j^2 f_j\left(\frac{n-1}{n}x\right) \bar{f}_n\left(-\frac{x}{n}\right) \quad (x \in \mathbf{R})$$

for any fixed $j < n$, where $\alpha_j, \alpha_n^j, \beta_j^1, \beta_j^2 \in \mathbf{R}_0$ are arbitrary constants with $\alpha_j \alpha_n^j \beta_j^1 \beta_j^2 = 1$ and the functions $a_j, a_n^j: \mathbf{R} \rightarrow \mathbf{R}$ and $n_j: \mathbf{R} \rightarrow \mathbf{R}$ satisfy the functional equations (1.7) and (1.8) respectively.

From (2.5) using (2.7) and (2.8), we get

$$f_n(x_n) = \frac{\alpha_n^j \alpha_j}{\prod_{i=1}^{n-1} \alpha_i} \exp \left[a_j(x_n) + a_n^j(x_n) - \sum_{i=1}^{n-1} a_i(x_n) + n n_j(x_n) - \sum_{i=1}^{n-1} n_i(x_n) \right]$$

for any fixed $j < n$ and for all $x_n \in \mathbf{R}$. Thus $n_j(x) = n(x)$ ($x \in \mathbf{R}$) ($j=1, \dots, n-1$). By $a_j(x) + a_n^j(x) = a_k(x) + a_n^k(x)$ ($x \in \mathbf{R}$) and $\alpha_n^j \alpha_j = \alpha_n^k \alpha_k$ the expression $a_j(x) + a_n^j(x) - \sum_{i=1}^{n-1} a_i(x)$ and $\frac{\alpha_n^j \alpha_j}{\sum_{i=1}^{n-1} \alpha_i}$ depend only on n . Denoting these by $a_n(x)$ and α_n respectively, we get that

$$(2.11) \quad f_n(x_n) = \alpha_n \exp[a_n(x_n) + n(x_n)] \quad (x_n \in \mathbf{R}).$$

(2.7) and (2.11) gives (2.1).

From (2.9) using (2.5) and (2.1), we get the representation (2.2) for g .

From (1) by $x_n = 0$ and using (2.1) and (2.2), we get (2.3).

It is easy to see that functions (2.1)–(2.3) satisfy the functional equation (1). From Theorem 2.1 we can easily obtain

Corollary 2.1. *If the measurable functions f_i , g and G satisfy the conditions of Theorem 2.1, then*

$$(2.12) \quad f_i(x_i) = \alpha_i \exp [a_i x_i + b x_i^2] \quad (x_i \in \mathbf{R}; i = 1, \dots, n),$$

$$(2.13) \quad g(x) = \beta \prod_{i=1}^n \alpha_i \exp \left[\frac{1}{n} \left(\sum_{i=1}^n a_i x + b x^2 \right) \right] \quad (x \in \mathbf{R}),$$

$$(2.14) \quad G(x_1, \dots, x_{n-1}) = \alpha_n \frac{\prod_{i=1}^{n-1} f_i(x_i)}{g\left(\sum_{i=1}^{n-1} x_i\right)} \quad (x_i \in \mathbf{R}; i = 1, \dots, n-1),$$

where $a_i, b \in \mathbf{R}$, $\alpha_i, \beta \in \mathbf{R}_0$ ($i=1, \dots, n$) are arbitrary constants.

PROOF. The functions $\bar{f}_n$ and $\bar{G}$ (defined by (2.5) and (2.6) respectively) are measurable. Thus the functions $f_j, \bar{f}_n, g$ and $\bar{G}$ satisfy the conditions of Corollary 1.1 if $a=1, b=n-1, c=1, d=-1$, therefore the functions $(\alpha_j, n_j$ in 2.7), (2.8) have the form

$$a_j(x_j) = a_j x_j, \quad n_j(x_j) = b_j x_j^2, \quad a_n^j(x_n) = a_n^j x_n \quad (x_j, x_n \in \mathbf{R})$$

for any fixed $j < n$, where $a_j, b_j, a_n^j \in \mathbf{R}$ are arbitrary constants. Then following the proof of Theorem 2.1, we get that

$$a_i(x_i) = a_i x_i \quad (x_i \in \mathbf{R}) \quad (i = 1, \dots, n); \quad n_j(x) = n(x) = b x^2 \quad (x \in \mathbf{R}; j = 1, \dots, n-1),$$

where $a_i, b \in \mathbf{R}$ are arbitrary constants and from (2.1)–(2.3) we can deduce the expressions (2.12)–(2.14) for f_i, g, G respectively.

3. On the functional equation (2)

Lemma 3.1. *Suppose that the functions f_i, g and G satisfy the functional equation (2) and there exist sets $T_i \subset \mathbf{R}_+$ ($i=1, \dots, n$) of positive Lebesgue-measure such that $\prod_{i=1}^n f_i(x_i) \neq 0$ for all $x_i \in T_i$ ($i=1, \dots, n$). Then $\prod_{i=1}^n f_i(x_i) g(u_1) G(u_2, \dots, u_n) \neq 0$ for all $x_i, u_i \in \mathbf{R}_+$ ($i=1, \dots, n$).*

PROOF. (2) implies that $g\left(\sum_{i=1}^n x_i\right) G\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) \neq 0$ for all $x_i \in T_i$ ($i=1, \dots, n$), that is

(3.1)

$$g(u_1) G(u_2, \dots, u_n) \neq 0 \quad \text{for} \quad u_1 \in \sum_{i=1}^n T_i, (u_2, \dots, u_n) \in \bigcup_{x_n \in T_n} \left[\frac{T_1}{x_n} \times \dots \times \frac{T_{n-1}}{x_n} \right].$$

Substituting

$$u_1 = \sum_{i=1}^n x_i, \quad u_{i+1} = \frac{x_i}{x_n} \quad (i = 1, \dots, n-1),$$

we get from (2) the equation

$$(3.2) \quad \prod_{i=1}^{n-1} f_i \left(\frac{u_1 u_{i+1}}{1 + \sum_{j=1}^{n-1} u_{j+1}} \right) f_n \left(\frac{u_1}{1 + \sum_{j=1}^{n-1} u_{j+1}} \right) = g(u_1) G(u_2, \dots, u_n) \quad (u_1, \dots, u_n \in \mathbf{R}_+),$$

which together with (3.1) implies that

$$\prod_{i=1}^n f_i(x_i) \neq 0 \quad \text{if} \quad \begin{cases} x_i \in A_i = \bigcup_{x_n \in T_n} \left(\frac{\left(\sum_{j=1}^n T_j \right) T_i}{x_n + \sum_{j=1, j \neq i}^n T_j + T_i} \right) & (i = 1, \dots, n-1) \\ x_n \in A_n = \bigcup_{x_n \in T_n} \left(\frac{\sum_{j=1}^n T_j}{1 + \sum_{j=1}^{n-1} \frac{T_j}{x_n}} \right). \end{cases}$$

By a theorem of Steinhaus (see [4]) the sets A_i ($i = 1, \dots, n$) contain intervals $I_i^{(0)} = [a_i^{(0)}, b_i^{(0)}] \subset A_i \subset \mathbf{R}_+$ ($i = 1, \dots, n$) such that

$$(3.3) \quad \prod_{i=1}^n f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in I_i^{(0)} \quad (i = 1, \dots, n).$$

Repeating this argument (using $I_i^{(0)}$ instead T_i), we have

$$(3.4) \quad \prod_{i=1}^n f_i(x_i) \neq 0, \quad \text{if} \quad \begin{cases} x_i \in B_i = \bigcup_{x_n \in I_n^{(0)}} \left(\frac{\sum_{j=1}^n I_j^{(0)} \frac{I_i^{(0)}}{x_n + \sum_{j=1, j \neq i}^n I_j^{(0)} + I_i^{(0)}} \right) & (i = 1, \dots, n-1) \\ x_n \in B_n = \bigcup_{x_n \in I_n^{(0)}} \left(\frac{\sum_{j=1}^n I_j^{(0)}}{1 + \sum_{j=1}^{n-1} \frac{I_j^{(0)}}{x_n}} \right). \end{cases}$$

It is easy to see that the sets B_i in (3.4) are the intervals

$$I_j^{(1)} = \left[\frac{\sum_{j=1}^n a_j^{(0)}}{a_i^{(0)} + \sum_{j=1, j \neq i}^n b_j^{(0)}} a_i^{(0)}, \frac{\sum_{j=1}^n b_j^{(0)}}{b_i^{(0)} + \sum_{j=1, j \neq i}^n a_j^{(0)}} b_i^{(0)} \right] \quad (i = 1, \dots, n).$$

Thus

$$(3.5) \quad \prod_{i=1}^n f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in I_i^{(1)} = [a_i^{(1)}, b_i^{(1)}] \quad (i = 1, \dots, n).$$

By induction, we get a sequence $I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}]$ ($i = 1, \dots, n; k = 0, 1, \dots$) with property

$$(3.6) \quad I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}] \supseteq \left[\left(\frac{\sum_{j=1}^n a_j^{(0)}}{a_i^{(0)} + \sum_{j=1, j \neq i}^n b_j^{(0)}} \right)^{k-1} a_i^{(0)}, \left(\frac{\sum_{j=1}^n b_j^{(0)}}{b_i^{(0)} + \sum_{j=1, j \neq i}^n a_j^{(0)}} \right)^{k-1} b_i^{(0)} \right]$$

$(i = 1, \dots, n)$

and

$$\prod_{i=1}^n f_i(x_i) \neq 0, \quad \text{if } x_i \in I_i^{(k)} \quad (i = 1, \dots, n; k = 0, 1, 2, \dots)$$

From (3.6) one can see that $a_i^{(k)} \rightarrow 0, b_i^{(k)} \rightarrow \infty$ as $k \rightarrow \infty$, therefore

$$\prod_{i=1}^n f_i(x_i) \neq 0, \quad \text{if } x_i \in \mathbf{R}_+ \quad (i = 1, \dots, n).$$

This and (2) gives that

$$g(u_1)G(u_2, \dots, u_n) \neq 0 \quad \text{if } u_i \in \mathbf{R}_+ \quad (i = 1, \dots, n),$$

which completes the proof of Lemma 3.1.

Now we can easily prove

Theorem 3.1. Suppose that the functions f_i, g and G satisfy the conditions of Lemma 3.1, then

$$(3.7) \quad f_i(x) = A_i \exp[a(x) + m_i(x)] \quad (x \in \mathbf{R}_+; i = 1, \dots, n),$$

$$(3.8) \quad g(x) = B \exp \left[a(x) + \sum_{i=1}^n m_i(x) \right] \quad (x \in \mathbf{R}_+),$$

$$(3.9) \quad G(x_1, \dots, x_{n-1}) = C \exp \left[\sum_{i=1}^{n-1} m_i \left(\frac{x_i}{1 + \sum_{i=1}^{n-1} x_i} \right) - m_n \left(1 + \sum_{i=1}^{n-1} x_i \right) \right]$$

$(x_i \in \mathbf{R}_+),$

where the functions $a: \mathbf{R} \rightarrow \mathbf{R}$ and $m_i: \mathbf{R}_+ \rightarrow \mathbf{R}$ ($i = 1, \dots, n$) fulfil the functional equations (1.7) and (1.19) respectively, $A_i, B, C \in \mathbf{R}_0$ are arbitrary constants with $\prod_{i=1}^n A_i = BC$.

PROOF. Using Lemma 3.1, it follows that $\prod_{i=1}^n f_i(x_i)g(u_1)G(u_2, \dots, u_n) \neq 0$ for all $x_i, u_i \in \mathbf{R}_+ \quad (i = 1, \dots, n)$.

Now, let $j < n$ be fixed and $x_i = x_n + x_j$ if $i \neq j, n$. Then we get from (2) that

$$f_n(x_n)f_j(x_j) \prod_{\substack{i=1 \\ i \neq j}}^{n-1} f_i(x_n + x_j) = g[(n-1)(x_n + x_j)]G \left(1 + \frac{x_j}{x_n}, \dots, \frac{x_j}{x_n}, \dots, 1 + \frac{x_j}{x_n} \right)$$

for all $x_n, x_j \in \mathbf{R}_+$. This implies the functional equation

$$(3.10) \quad f_n(x_n) f_j(x_j) = p(x_n + x_j) q\left(\frac{x_j}{x_n}\right) \quad (x_n, x_j \in \mathbf{R}_+),$$

where

$$(3.11) \quad p(u) = \frac{g[(n-1)u]}{\prod_{\substack{i=1 \\ i \neq j}}^{n-1} f_i(u)} \quad (u \in \mathbf{R}_+)$$

and

$$(3.12) \quad q(v) = G(1+v, \dots, \frac{v}{j}, \dots, 1+v) \quad (v \in \mathbf{R}_+).$$

Thus the functions f_n, f_j, p, q satisfy the condition of Theorem 1.2. Therefore f_n and f_j are of the forms

$$(3.13) \quad f_i(x_i) = A_i \exp[a(x) + m_i(x)] \quad (x \in \mathbf{R}_+; i = j, n)$$

for any fixed $j < n$, where the functions $a: \mathbf{R} \rightarrow \mathbf{R}$ and $m_i: \mathbf{R}_+ \rightarrow \mathbf{R}$ ($i = n, j$) satisfy the functional equations (1.7) and (1.19) respectively, $A_i \in \mathbf{R}_0$ ($i = n, j$) are arbitrary constants. Hence (3.7) holds.

From (2) by $x_i = x$ ($i = 1, \dots, n$) we get

$$\prod_{i=1}^n f_i(x) = g(nx) G(1, \dots, 1) \quad (x \in \mathbf{R}_+),$$

which together with (3.7) implies (3.8), where $B \in \mathbf{R}_0$ is an arbitrary constant.

Finally from (2) by $x_n = 1$ and using (3.7) and (3.8), we get (3.9) for G , where C is an arbitrary constant.

It is easy to see that the functions (3.7)–(3.9) satisfy the functional equation (2) indeed if $\prod_{i=1}^n A_i = BC$.

Now, we can easily obtain

Corollary 3.1. *If the measurable functions f_i, g and G satisfy the conditions of Theorem 3.1, then*

$$(3.14) \quad f_i(x) = A_i \exp[ax + b_i \ln x] \quad (x \in \mathbf{R}_+, i = 1, \dots, n),$$

$$(3.15) \quad g(x) = B \exp\left[ax + \left(\sum_{i=1}^n b_i\right) \ln x\right] \quad (x \in \mathbf{R}_+),$$

$$(3.16) \quad G(x_1, \dots, x_{n-1}) = C \exp\left[\sum_{i=1}^{n-1} b_i \ln\left(\frac{x_i}{1 + \sum_{i=1}^{n-1} x_i}\right) - b_n \ln\left(1 + \sum_{i=1}^{n-1} x_i\right)\right] \\ (x_i \in \mathbf{R}_+),$$

where $a, b_i \in \mathbf{R}$; $A_i, B, C \in \mathbf{R}_0$ ($i = 1, \dots, n$) are arbitrary constants with $\prod_{i=1}^n A_i = BC$.

The proof of this corollary is similar to that of Corollary 2.1. (Here we need Corollary 1.2 instead of Corollary 1.1.)

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