

A minimax theorem for additive functions

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1. Let $g(n)$ be a nonnegative additive function, assume that tends to zero monotonically on the set of primes. Let

$$(1.1) \quad \psi(y) = \sum_{p \leq y} g(p),$$

$$(1.2) \quad \beta(p) = \sup_{\alpha \geq 1} g(p^\alpha),$$

$$(1.3) \quad \sup_{\alpha \geq 1} |g(p^\alpha) - g(p)| \leq t(p).$$

Assume that

$$(1.4) \quad \psi(2y) - \psi(y) = o(1) \quad (y \rightarrow \infty)$$

$$(1.5) \quad \psi(y) \rightarrow \infty,$$

$$(1.6) \quad \sum_p t(p) < \infty.$$

Let $k \geq 1$,

$$(1.7) \quad E_k(x) = \max_{n \leq x} \min_{j=1, \dots, k} g(n+j).$$

We should like to determine $E_k(x)$ as $x \rightarrow \infty$ for every fixed k . For every integer n let

$$(1.8) \quad A_k(n) = \prod_{\substack{p^\alpha \parallel n \\ p \leq k}} p^\alpha$$

$$(1.9)-(1.10) \quad B_k = \sup_{n \geq 1} \frac{1}{k} \left\{ \sum_{j=1}^k g(A_k(n+j)) \right\}, \quad C_k = \frac{1}{k} \sum_{p \geq k} (\beta(p) - g(p)).$$

We shall prove the following

Theorem 1. *On the assumptions stated above*

$$(1.11) \quad \lim_{x \rightarrow \infty} \left(E_k(x) - \frac{\psi(\log x)}{k} \right) = B_k + C_k - \frac{\psi(k)}{k}.$$

Supposing furthermore that $g(p) \leq g(p^2) \leq g(p^3) \leq \dots$, we have

$$(1.12) \quad B_k = \frac{1}{k} \sum_{p \leq k} \varrho_{p,k},$$

where

$$(1.13) \quad \varrho_{p,k} = \beta(p) + \sum_{r=1}^{\infty} g(p^r) \left\{ \left[\frac{k-1}{p^r} \right] - \left[\frac{k-1}{p^{r+1}} \right] \right\}.$$

Remarks. 1. Assuming that $g(n)$ is strongly additive, i.e. $g(p^\alpha) = g(p)$ for $\alpha \geq 2$, we have

$$(1.14) \quad \varrho_{p,k} = \begin{cases} \left(1 + \left[\frac{k}{p}\right]\right) g(p) & \text{if } p+k, \\ \frac{k}{p} g(p) & \text{if } p|k. \end{cases}$$

2. Putting $g(n) = \log \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of the divisors of n we have

$$(1.15) \quad L(k) = \overline{\lim}_{n \rightarrow \infty} (\log \log n)^{1/k} \min \left(\frac{\sigma(n+1)}{n+1}, \dots, \frac{\sigma(n+k)}{n+k} \right)$$

where

$$(1.16) \quad L(k) = \left(\frac{6}{\pi^2} e^\gamma \right)^{1/k} \exp \left(B_k + C_k - \frac{\psi(k)}{k} \right),$$

$$(1.17) \quad \psi(k) = \sum_{p \leq k} \log \left(1 + \frac{1}{p} \right), \quad C_k = \frac{1}{k} \sum_{p > k} \log \frac{1}{1 - 1/p^2}$$

and B_k is computed from (1.12), (1.13) by putting $\beta(p) = \log(1 - 1/p)^{-1}$, $g(p^r) = \log(1 + p^{-1} + \dots + p^{-r})$.

The proof of (1.14) is an obvious consequence of (1.13). To prove (1.15) we have to observe only that

$$\prod_{p \leq y} (1 + 1/p) = (1 + o(1)) \frac{6}{\pi^2} e^\gamma \log y.$$

3. Putting $g(n) = -\log \frac{\varphi(n)}{n}$, our theorem gives the recent result of M. HAUSMAN [1].

2. Proof of the assertion

Let

$$g_y(n) = \sum_{\substack{p^\alpha \parallel n \\ p^\alpha \leq y}} g(p^\alpha); \quad g(n; y) = \sum_{\substack{p^\alpha \parallel n \\ p^\alpha > y}} g(p^\alpha).$$

First we observe that $g(n; \log n) \rightarrow 0$ as $n \rightarrow \infty$. Indeed, if $q_1 < q_2 < \dots < q_r$ are the sequence of all prime divisors of n that are greater than $\log n$, then from the monotonicity of $g(p)$ we have

$$g(n; \log n) = \sum_{i=1}^r \beta(q_i) \leq \sum_{i=1}^r g(p_i) + \sum_{p > y} t(p)$$

where $p_1 < p_2 < \dots < p_r$ the smallest r primes greater than $\log n$. Since

$$(\log n)^r \equiv \prod_{i=1}^r p_i \equiv \prod_{i=1}^r q_i \equiv n,$$

we have $r \equiv (\log n)(\log \log n)^{-1}$, and so $p_r \equiv 4 \log n$.

So, by (1.4) we have

$$\sum_{i=1}^r g(p_i) \equiv \psi(4 \log n) - \psi(\log n) \rightarrow 0.$$

Now we give an appropriate upperestimation for $E_k(x)$. It is obvious that

$$E_k(x) \equiv \max_{n \leq x} \{k^{-1} \{g(n+1) + \dots + g(n+k)\}\}.$$

From (1.9) we have

$$E_k(x) \equiv B_k + \max_{n \leq x} \left\{ k^{-1} \sum_{j=1}^k g(n+j; k) \right\}.$$

Since for $p > k$ p divides $n+j$ at most once and $g(n+j, \log x) = \sigma(1)$, we have

$$k^{-1} \sum_{j=1}^k g(n+j; k) \equiv k^{-1} \sum_{k < p \leq \log x} \beta(p) = C_k + k^{-1} (\psi(\log x) - \psi(k)),$$

whence in (1.11) the inequality $\equiv$ holds.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be arbitrary small positive constants. Let n_0 be chosen so that

$$(2.1) \quad k^{-1} \sum_{j=1}^k g(A_k(n_0+j)) \equiv B_k - \varepsilon_1.$$

Let $P^{(1)} = \prod_{p \leq k} p$,

$$P^{(2)} = \text{l.c.m.} \{A_k(n_0+j) \mid j = 1, \dots, k\}.$$

It is obvious that $A_k(n+j) = A_k(b_0+j)$, $j \in [1, k]$ for $n \equiv n_0 \pmod{P^{(1)}P^{(2)}}$. Let y be so large and the exponents β_j for the primes p_j in the interval (k, y) be chosen so that

$$(2.2) \quad \Sigma g(p_j^{\beta_j}) \equiv kC_k + (\psi(y) - \psi(k)) - \varepsilon_2.$$

Let $P_3 = \prod p_j^{\beta_j}$. Let now

$$P = \prod_{y < p \leq \frac{1}{2} \log x} p,$$

$Q_1, Q_2, \dots, Q_k$ be the products of primes from $(y, \frac{1}{2} \log x)$, so that $P = Q_1 \dots Q_k$

The congruence system

$$n \equiv n_0 \pmod{P^{(1)}P^{(2)}}, \quad n+1 \equiv 0 \pmod{P^{(3)}},$$

$$n+1 \equiv 0 \pmod{Q_1}, \quad n+j \equiv 0 \pmod{Q_j} \quad (j = 2, \dots, k)$$

has a solution in the interval $[1, P^{(1)}P^{(2)}P^{(3)}P]$. From the nonnegativity of g we have

$$(2.3) \quad \begin{cases} g(n+1) \equiv \gamma_1 + \delta + g(Q_1) \\ g(n+j) \equiv \gamma_j + g(Q_j) \quad (j = 2, \dots, k), \end{cases}$$

where

$$\gamma_j = g(A_k(n_0 + j)), \quad \delta = g(P^{(3)}).$$

We have

$$\psi\left(\frac{1}{2}\log x\right) - \psi(y) = g(P) = g(Q_1) + \dots + g(Q_k).$$

Let $A = \psi\left(\frac{1}{2}\log x\right) - \psi(y) + (\gamma_1 + \dots + \gamma_k + \delta)$. Since $g(p) \rightarrow 0$ for $p \rightarrow \infty$, we can choose a partition of the primes in $\left(y, \frac{1}{2}\log x\right)$ such that

$$(2.4) \quad \left|g(Q_1) + (\gamma_1 + \delta) - \frac{A}{k}\right| \leq \varepsilon_3, \quad \left|g(Q_j) + \gamma_j - \frac{A}{k}\right| \leq \varepsilon_3.$$

Taking into account (2.1), (2.2) we get

$$k^{-1}(\gamma_1 + \dots + \gamma_k + \delta) \geq B_k + C_k + k^{-1}(\psi(y) - \psi(k)) - k^{-1}(\varepsilon_1 + \varepsilon_2),$$

and from (2.3), (2.4) we deduce

$$\min_{j=1, \dots, k} g(n+j) \geq B_k + C_k + k^{-1} \left(\psi\left(\frac{1}{2}\log x\right) - \psi(k) \right) - k^{-1}(\varepsilon_1 + \varepsilon_2) - \varepsilon_3.$$

Since $\psi(\log x) - \psi\left(\frac{1}{2}\log x\right) = \sigma(1)$, we get the desired result.

Reference

- [1] M. HAUSMAN, Generalization of a theorem of Landau, *Pacific J. Math.* **84** (1979), 91—95.

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