On some properties of linear recurrences

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1. Introduction

A linear recurrence $G=\{G_n\}_{n=0}^{\infty}$ of order k (>1) is defined by rational integers $A_1, A_2, \ldots, A_k$ and by recursion $G_n=A_1G_{n-1}+A_2G_{n-2}+\ldots+A_kG_{n-k}$ $(n\geqq k)$, where the initial values $G_0, G_1, \ldots, G_{k-1}$ are fixed not all zero rational integers and $A_k\neq 0$. We suppose that the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ of the characteristic polynomial

$$g(x) = x^k - A_1x^{k-1} - \ldots - A_k$$

are distinct and so

$$D = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix} = \prod_{1\leqq i<j\leqq k} (\alpha_j-\alpha_i) \neq 0.$$

Let $x_0, x_1, \ldots, x_{k-1}$ be variables. We define a $k\times k$ determinant M_i from D by rejecting the i-th column $(i=1, 2, \ldots, k)$ and inscribing $x_0, x_1, \ldots, x_{k-1}$ as first column. Thus

$$M_i = \begin{vmatrix} x_0 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ x_1 & \alpha_1 & \ldots & \alpha_{i-1} & \alpha_{i+1} & \ldots & \alpha_k \\ x_2 & \alpha_1^2 & \ldots & \alpha_{i-1}^2 & \alpha_{i+1}^2 & \cdots & \alpha_k^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{k-1} & \alpha_1^{k-1} & \ldots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix}.$$

Furthermore we define a polynomial $f(x_0, x_1, \ldots, x_{k-1})$ by

$$f(x_0, x_1, \ldots, x_{k-1}) = D^{2-k} \prod_{i=1}^{k} M_i.$$

In case $k=2$

$$f(x_0, x_1) = (x_0\alpha_1 - x_1)(x_0\alpha_2 - x_1)$$

and if $g(x)$ is irreducible over the rationals and $K=\mathbf{Q}(\alpha_1)$ is a quadratic number field then it can be written in the form

$$f(x_0, x_1) = N_{K/\mathbf{Q}}(\xi) = N(\xi),$$

where $\xi=x_0\alpha_1-x_1$. On the other hand

$$f(x_0, x_1) = x_1^2-(\alpha_1+\alpha_2)x_1x_0+\alpha_1\alpha_2x_0^2 = x_1^2-A_1x_1x_0-A_2x_0^2$$

and one can easily check by induction that

$$f(G_n, G_{n+1}) = C(-A_2)^n$$

for any integer $n(\geqq 0)$, where $C=f(G_0, G_1)$. Thus we have

$$N(\xi) = C(-A_2)^n$$

for $\xi=G_n\alpha_1-G_{n+1}$. It is a well-known identity in the form

$$G_{n+1}^2-A_1G_{n+1}G_n-A_2G_n^2 = C(-A_2)^n$$

which was proved first by E. LUCAS for case $G_0=0$, $G_1=1$ (see L. E. DICKSON [1], p. 396).

In the following we extend this result to arbitrary integer k and give some applications of it.

2. Results

Let $k\geqq 2$ be an integer. Using the previous notations we prove:

Theorem 1. *The polynomial*

$$f(x_0, x_1, \ldots, x_{k-1}) = D^{2-k}\prod_{i=1}^{k} M_i$$

in variables $x_0, x_1, \ldots, x_{k-1}$ *has rational integer coefficients and the coefficient of* x_{k-1}^k *is one. Furthermore*

$$f(G_n, G_{n+1}, \ldots, G_{n+k-1}) = F_0[(-1)^{k-1}A_k]^n$$

for all integer $n\geqq 0$, *where*

$$F_0 = f(G_0, G_1, \ldots, G_{k-1}).$$

Let the characteristic polynomial $g(x)$ of the sequence G be irreducible over the rationals with roots $\alpha=\alpha_1, \alpha_2, \ldots, \alpha_k$. In this case $K=\mathbf{Q}(\alpha)$ is an algebraic number field of degree k and the algebraic numbers $1, \alpha, \alpha^2, \ldots, \alpha^{k-1}$ are linearly independent over the field of rational numbers, furthermore the numbers

$$\xi = a_0+a_1\alpha+\ldots+a_{k-1}\alpha^{k-1} \tag{1}$$

with rational integer a_i's are algebraic integer in K and the set of ξ's is an order of K. The following statement is true.

Theorem 2. *If the* a_i*'s* $(i=0, 1, \ldots, k-1)$ *in* (1) *have the form*

$$a_i = G_{n+k-i-1}-\sum_{j=1}^{k-i-1} A_jG_{n+k-i-j-1},$$

where n is an integer, then the norm of ξ is

$$N(\xi) = N(\xi_0) \cdot [(-1)^{k-1} A_k]^n, \tag{2}$$

where ξ_0 is a special case of ξ determining a_i's by $n=0$.

Norm form equation (2) can be used to express the units of algebraic number fields. A simple case is the cubic field with negative discriminant. In this case, if $K=\mathbf{Q}(\alpha)$, among the conjugates of α there are one real and two complex algebraic number and the number of the fundamental units in K is one. Let ω be one of the fundamental units. ω is an algebraic integer of degree three and so $K=\mathbf{Q}(\alpha)=\mathbf{Q}(\omega)$. Furthermore K does not contain any roots of unity except ± 1. Thus all units of K are of the form $\varepsilon=\pm\omega^n$. For the units of K we prove:

Theorem 3. *Let $K=\mathbf{Q}(\omega)$ be an algebraic number field of degree three with negative discriminant. Suppose ω is a fundamental unit in K with minimal defining polynomial*

$$h(x) = x^3 - Ax^2 - Bx - C,$$

where $|C|=1$. Let $G=\{G_n\}_{n=-\infty}^{+\infty}$ be a linear recurrence of order three defined by initial terms $G_0=G_1=0$, $G_2=1$ and by recursion

$$G_n = AG_{n-1} + BG_{n-2} + CG_{n-3},$$

which is also defined for negative subscripts since

$$G_n = (G_{n+3} - AG_{n+2} - BG_{n+1})/C$$

is an integer by $|C|=1$. Then the all units in K are the numbers

$$\varepsilon = \pm[G_n\omega^2 + (G_{n+1} - AG_n)\omega + (G_{n+2} - AG_{n+1} - BG_n)] = \pm\varepsilon_n,$$

where n runs over the integers.

We note that C. KLIORYS [5] also has given a connection between the units of some algebraic number fields and linear recurrences. He showed that in algebraic number field $Q(\omega)$ of degree $2k$, where ω is a root of polynomial $x^{2k}-x^k-1$, the powers of ω can be expressed in the form $\omega^n=a_0+a_1\omega+\ldots+a_{2k-1}\omega^{2k-1}$, where the a_i's are Fibonacci numbers.

In [4] we gave a diophantine representation of the terms of generalized Fibonacci sequence which was the generalization of the results of J. P. JONES [2, 3] on Fibonacci and Lucas sequences. These results can be extended to some third order linear recursive sequences, too.

Theorem 4. *Let $K=\mathbf{Q}(\omega)$ and $G=\{G_n\}_{n=-\infty}^{+\infty}$ be an algebraic number field of degree three and a third order linear recurrence respectively defined in Theorem 3 and let $f(x_0, x_1, x_2)$ be the polynomial defined in Theorem 1. Then the set of the nonnegative values of function*

$$F(x, y, z) = [2 - (f(x, y, z))^2] \cdot |x|$$

is the set of the absolute values of the terms of sequence G, as x, y, and z run over the integers.

3. A lemma

In order to prove the theorems we require a lemma.

Let D be the $k\times k$ determinant defined in section 1 and let $D_{i,j}^{(k)}$ be the $(k-1)\times(k-1)$ determinant obtained from D by rejection of the i-th column and the j-th row $(1\leqq i, j\leqq k)$. Thus

$$D_{i,j}^{(k)}=\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{i-1} & \alpha_{i+1} & \dots & \alpha_k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{j-2} & \alpha_2^{j-2} & \dots & \alpha_{i-1}^{j-2} & \alpha_{i+1}^{j-2} & \dots & \alpha_k^{j-2} \\ \alpha_1^{j} & \alpha_2^{j} & \dots & \alpha_{i-1}^{j} & \alpha_{i+1}^{j} & \dots & \alpha_k^{j} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix}.$$

In case $j=k$ we denote $D_{i,j}^{(k)}$ by $D_i^{(k)}$, i.e.

$$D_i^{(k)}=D_{i,j}^{(k)}.$$

Using this notations, we prove:

Lemma. *For every indices* i, j $(1\leqq i, j\leqq k)$

$$D_{i,j}^{(k)}=D_i^{(k)}\cdot S_i^{(k-j)}, \tag{3}$$

where $S_i^{(k-j)}$ *denotes the elementary symmetrical polynomial of degree* $k-j$ *of variables* $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k$ *for* $k-j>0$ *and* $S_i^{(0)}=1$.

Proof. The statement is obviously true in cases $j=1$ and $j=k$, furthermore it can be easily seen for $k=2$ and $k=3$ with arbitrary i and j. Thus we can suppose that $i>1$, $1<j<k$, $k>3$ and it is enough to prove that

$$D_{i,j}^{(k-1)}=D_i^{(k-1)}\cdot S_i^{(k-1-j)}$$

implies equation (3).

Subtracting the first column from the others in determinant $D_{i,j}^{(k)}$, we get $1, 0, 0, \dots, 0$ as first row and so $D_{i,j}^{(k)}$ is equal to a $(k-2)\times(k-2)$ determinant with row-vectors

$$v_t=(\alpha_2^t-\alpha_1^t, \alpha_3^t-\alpha_1^t, \dots, \alpha_{i-1}^t-\alpha_i^t, \alpha_{i+1}^t-\alpha_1^t, \dots, \alpha_k^t-\alpha_1^t)$$

$(1\leqq t\leqq k-1, t\neq j-1)$. The column-vectors of this determinant are divisible by $\alpha_2-\alpha_1$, $\alpha_3-\alpha_1, \dots, \alpha_{i-1}-\alpha_1, \alpha_{i+1}-\alpha_1, \dots$ and $\alpha_k-\alpha_1$ respectively, therefore

$$D_{i,j}^{(k)}=D'\prod_{\substack{2\leqq r\leqq k\\ r\neq i}}(\alpha_r-\alpha_1), \tag{4}$$

where D' is a $(k-2)\times(k-2)$ determinant. The coordinates of the row-vectors $v_t'=(c_{t,2}, c_{t,3}, \dots, c_{t,i-1}, c_{t,i+1}, \dots, c_{t,k})$ of determinant D' are

$$c_{t,r}=\alpha_r^{t-1}+\alpha_r^{t-2}\alpha_1+\dots+\alpha_r\alpha_1^{t-2}+\alpha_1^{t-1}$$

$(2\leqq r\leqq k,\ r\neq i)$ and so

$$v_t'-\alpha_1 v_{t-1}'=(\alpha_2^{t-1}, \alpha_3^{t-1}, \dots, \alpha_{i-1}^{t-1}, \alpha_{i+1}^{t-1}, \dots, \alpha_k^{t-1})$$

for $1<t\neq j$ and

$$v'_j-\alpha_1^2 v'_{j-2}=(\alpha_2^{j-1}+\alpha_2^{j-2}\alpha_1,\ \alpha_3^{j-1}+\alpha_3^{j-2}\alpha_1,\ \ldots$$
$$\ldots,\ \alpha_{i-1}^{j-1}+\alpha_{i-1}^{j-2}\alpha_1,\ \alpha_{i+1}^{j-1}+\alpha_{i+1}^{j-2}\alpha_1,\ \ldots,\ \alpha_k^{j-1}+\alpha_k^{j-2}\alpha_1)$$

if $j\neq 2$. The first row of D' is $v'_1=(1, 1, \ldots, 1)$ in case $j\neq 2$ and $v'_1=(\alpha_2+\alpha_1, \alpha_3+\alpha_1, \ldots, \alpha_k+\alpha_1)$ in case $j=2$. From these follows, using the elementary properties of determinants and our notations, that

$$D'=D_{i-1,j-1}^{(k-1)}+\alpha_1\cdot D_{i-1,j}^{(k-1)}$$

and the determinants on right are constructed by $\alpha_2, \alpha_3, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k$. By our supposition we have

$$D'=D_{i-1}^{(k-1)}\cdot S_i'^{(k-j)}+\alpha_1\cdot D_{i-1}^{(k-1)}\cdot S_i'^{(k-1-j)} \tag{5}$$

where $S_i'^{(t)}$ is the elementary symmetrical polynomial of degree t of variables $\alpha_2, \alpha_3, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k$.

But

$$D_{i-1}^{(k-1)}\prod_{\substack{2\leqq r\leqq k\\ r\neq i}}(\alpha_r-\alpha_1)=D_i^{(k)}$$

and

$$S_i'^{(k-j)}+\alpha_1\cdot S_i'^{(k-1-j)}=S_i^{(k-j)}$$

so by (4) and (5) we get

$$D_{i,j}^{(k)}=D_i^{(k)}\cdot S_i^{(k-j)}.$$

Thus the Lemma is true for every integer $k\geqq 2$.

4. Proofs of the theorems

Proof of Theorem 1. The discriminant M_i is a linear form of variables $x_0, x_1, \ldots, x_{k-1}$ and, by the Lemma, the coefficient of x_{j-1} $(1\leqq j\leqq k)$ is

$$(-1)^{j-1}\cdot D_{i,j}^{(k)}=(-1)^{j-1}\cdot D_i^{(k)}\cdot S_i^{(k-j)}.$$

From this follows, using the identy

$$\prod_{i=1}^{k} D_i^{(k)}=D^{k-2},$$

that the coefficient of $x_0^{r_0}\cdot x_1^{r_1}\ldots x_{k-1}^{r_{k-1}}$ $(0\leqq r_i\leqq k,\ r_0+r_1+\ldots+r_{k-1}=k)$ in the polynomial $f(x_0, x_1, \ldots, x_{k-1})$ is

$$(-1)^q\cdot Q(r_0, r_1, \ldots, r_{k-1}),$$

where $q=0\cdot r_0+1\cdot r_1+2\cdot r_2+\ldots+(k-1)\cdot r_{k-1}$ and the value of $Q(r_0, r_1, \ldots, r_{k-1})$ is

$$\Sigma\left(S_{i_1}^{(k-1)}\cdot S_{i_2}^{(k-1)}\ldots S_{i_{r_0}}^{(k-1)}\right)\ldots\left(S_{t_1}^{(0)}\cdot S_{t_2}^{(0)}\ldots S_{t_{r_{k-1}}}^{(0)}\right).$$

The summation is extended to such permutations $i_1, i_2, \ldots, i_{r_0}, \ldots, t_1, t_2, \ldots, t_{r_{k-1}}$ of elements 1, 2, ..., k that the elements $(i_1, i_2, \ldots, i_{r_0})$, ... and $(t_1, t_2, \ldots, t_{r_{k-1}})$ respectively are not permutated among themselves. The sum is a symmetrical polynomial

in variables $\alpha_1, \alpha_2, \ldots, \alpha_k$, therefore $f(x_0, x_1, \ldots, x_{k-1})$ really has integer coefficients and the coefficient of x_{k-1}^k is

$$(-1)^q \cdot Q(0, 0, \ldots, 0, k) = S_1^{(0)} \cdot S_2^{(0)} \ldots S_k^{(0)} = 1$$

since $S_i^{(0)} = 1$ and $q = (k-1) \cdot k$ is an even integer.

We have to prove yet the identity

$$f(G_n, G_{n+1}, \ldots, G_{n+k-1}) = F_0[(-1)^{k-1} A_k]^n.$$

It is enough to prove that

$$f(G_{n+1}, G_{n+2}, \ldots, G_{n+k}) = (-1)^{k-1} A_k \cdot f(G_n, G_{n+1}, \ldots, G_{n+k-1})$$

for any $n \geqq 0$, since the statement is trivial for $n = 0$.

Let $M_i^{(t)}$ denote the $k \times k$ determinant which we obtain from M_i replacing the element $x_0, x_1, \ldots, x_{k-1}$ by $G_t, G_{t+1}, \ldots, G_{t+k-1}$ respectively. Let us consider the determinant

$$M_i^{(n+1)} = \begin{vmatrix} G_{n+1} & 1 & \ldots & 1 & 1 & \ldots & 1 \\ G_{n+2} & \alpha_1 & \ldots & \alpha_{i-1} & \alpha_{i+1} & \ldots & \alpha_k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ G_{n+k} & \alpha_1^{k-1} & \ldots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix}$$

and let us subtract the first, second, ... and $(k-1)$-th row multiplied by $A_{k-1}, A_{k-2}, \ldots$ and A_1 respectively from the k-th row. Since

$$G_{n+k} - A_1 G_{n+k-1} - \ldots - A_{k-2} G_{n+2} - A_{k-1} G_{n+1} = A_k G_n$$

by the definition of sequence G and

$$\alpha_j^{k-1} - A_1 \alpha_j^{k-2} - \ldots - A_{k-2} \alpha_j - A_{k-1} =$$

$$= \frac{1}{\alpha_j} (\alpha_j^k - A_1 \alpha_j^{k-1} - \ldots - A_{k-1} \alpha_j - A_k) + \frac{A_k}{\alpha_j} = \frac{A_k}{\alpha_j}$$

for $j = 1, 2, \ldots, k$ by the definition of α_j, furthermore

$$A_k = (-1)^{k+1} \alpha_1 \alpha_2 \ldots \alpha_k, \tag{6}$$

we have

$$M_i^{(n+1)} = A_k \begin{vmatrix} G_{n+1} & 1 & \ldots & 1 \\ G_{n+2} & \alpha_1 & \ldots & \alpha_k \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \ldots & \alpha_k^{k-1} \\ G_n & 1/\alpha_1 & \ldots & 1/\alpha_k \end{vmatrix} = (-1)^{k+1} \alpha_i \begin{vmatrix} G_{n+1} & \alpha_1 & \ldots & \alpha_k \\ G_{n+2} & \alpha_1^2 & \ldots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \ldots & \alpha_k^{k-1} \\ G_n & 1 & \ldots & 1 \end{vmatrix} =$$

$$= (-1)^{k+1} \cdot (-1)^{k-1} \alpha_i \begin{vmatrix} G_n & 1 & \ldots & 1 \\ G_{n+1} & \alpha_1 & \ldots & \alpha_k \\ G_{n+2} & \alpha_1^2 & \ldots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix} = \alpha_i M_i^{(n)}.$$

Thus, using identity (6), we get

$$f(G_{n+1}, G_{n+2}, \ldots, G_{n+k}) = D^{2-k} \cdot \prod_{i=1}^{k} M_i^{(n+1)} = D^{2-k} \cdot \prod_{i=1}^{k} \alpha_i M_i^{(n)} =$$

$$= \alpha_1 \cdot \alpha_2 \ldots \alpha_k \cdot D^{2-k} \cdot \prod_{i=1}^{k} M_i^{(n)} = (-1)^{k-1} A_k \cdot f(G_n, G_{n+1}, \ldots, G_{n+k-1}),$$

which proves the last statement of Theorem 1.

PROOF OF THEOREM 2. First we determine the elementary symmetrical polynomial $S_i^{(t)}$ $(0 \leqq t \leqq k-1)$ by α_i and by the elementary symmetrical polynomials A_1, $-A_2, \ldots, (-1)^{k+1} A_k$ of variables $\alpha_1, \alpha_2, \ldots, \alpha_k$. On account of symmetry we consider only the case $i=1$. If $t=0$ then, by the definition, $S_1^{(0)}=1$, furthermore obviously

$$S_1^{(1)} = A_1 - \alpha_1 = -(\alpha_1 - A_1)$$

and

$$S_1^{(2)} = -A_2 - \alpha_1 S_1^{(1)} = \alpha_1^2 - A_1 \alpha_1 - A_2.$$

In general we have

$$S_1^{(t)} = (-1)^t \cdot (\alpha_1^t - A_1 \alpha_1^{t-1} - A_2 \alpha_1^{t-2} - \ldots - A_{t-1} \alpha_1 - A_t) \tag{7}$$

since if (7) is true for $t=r-1$ $(\geqq 1)$, then

$$S_1^{(r)} = (-1)^{r+1} A_r - \alpha_1 S_1^{(r-1)} = (-1)^{r+1} A_r - (-1)^{r-1} \alpha_1 (\alpha_1^{r-1} - A_1 \alpha_1^{r-2} - \ldots - A_{r-1}) =$$

$$= (-1)^r \cdot (\alpha_1^r - A_1 \alpha_1^{r-1} - \ldots - A_{r-1} \alpha_1 - A_r).$$

Using (7) and the results obtained in the proof of Theorem 1, we get

$$M_1 = \sum_{j=1}^{k} (-1)^{j-1} D_1^{(k)} S_1^{(k-j)} x_{j-1} =$$

$$= (-1)^{k-1} D_1^{(k)} \sum_{j=1}^{k} (\alpha_1^{k-j} - A_1 \alpha_1^{k-j-1} - \ldots - A_{k-j-1} \alpha_1 - A_{k-j}) x_{j-1} =$$

$$= (-1)^{k-1} D_1^{(k)} \sum_{i=0}^{k-1} \left(x_{k-i-1} - \sum_{j=1}^{k-i-1} A_j x_{k-i-j-1} \right) \alpha_1^i.$$

If $x_0, x_1, \ldots, x_{k-1}$ are rational integers then

$$\xi = \sum_{i=0}^{k-1} \left(x_{k-i-1} - \sum_{j=1}^{k-i-1} A_j x_{k-i-j-1} \right) \alpha_1^i$$

is an algebraic integer in the number field $K=\mathbf{Q}(\alpha_1)$. Denote the conjugates of ξ by $\xi^{(1)}=\xi, \xi^{(2)}, \ldots, \xi^{(k)}$. Now, by the definition of the polynomial f, we get

$$f(x_0, x_1, \ldots, x_{k-1}) = D^{2-k} \cdot \prod_{i=1}^{k} M_i =$$

$$= D^{2-k} \cdot \prod_{i=1}^{k} [(-1)^{k-1} D_i^{(k)} \xi^{(i)}] = D^{2-k} \cdot D^{k-2} \cdot \prod_{i=1}^{k} \xi^{(i)} = N(\xi)$$

and from this, by Theorem 1, the statement follows.

PROOF OF THEOREM 3. We can use the previous results taking into our consideration that in this case $k=3$, $\alpha_1=\omega=\omega_1$, $\alpha_2=\omega_2$, $\alpha_3=\omega_3$, $A_1=A$, $A_2=B$, and $A_3=C$, where $\omega_1, \omega_2, \omega_3$ are the roots of the polynomial $h(x)$.

By the results, obtained in the proof of Theorem 2, we have

$$M_1^{(n)} = D_1^{(3)} \cdot \varepsilon_n,$$

where $\varepsilon_n=G_n\omega^2+(G_{n+1}-AG_n)\omega+(G_{n+2}-AG_{n+1}-BG_n)$. We have seen in the proof of Theorem 1 that

$$M_1^{(n+1)} = \omega M_1^{(n)}$$

which implies the equality

$$M_1^{(n)} = \omega^n \cdot M_1^{(0)}.$$

But

$$D_1^{(3)} = M_1^{(0)} = \begin{vmatrix} 1 & 1 \\ \omega_2 & \omega_3 \end{vmatrix},$$

and so

$$\varepsilon_n = \omega^n,$$

which proves the statement.

PROOF OF THEOREM 4. First we show that $f(x, y, z)=0$ for rational integers x, y, z if and only if $x=y=z=0$. If $f(x, y, z)=0$, then

$$M_i = D_i^{(3)}[x\omega_i^2+(y-Ax)\omega_i+(z-Ay-Bx)] = 0 \tag{8}$$

for some i ($i=1, 2$ or 3). But $D_i^{(3)}\neq 0$, since the roots of $h(x)$ are distinct, and ω_i is an algebraic integer of degree three, therefore (8) implies the equations

$$x = y-Ax=z-Ay-Bx = 0,$$

thus $x=y=z=0$. If $x=y=z=0$ then obviously $f(x, y, z)=0$.

The polynomial $2-(f(x, y, z))^2$ for integer x, y, z (not all zero) is non-negative if and only if

$$f(x, y, z) = \pm 1. \tag{9}$$

We have seen in the proof of Theorem 2 that

$$f(x, y, z) = N(x\omega^2+(y-Ax)\omega+(z-Ay-Bx)) = N(\xi)$$

where $\xi=x\omega^2+(y-Ax)\omega+(z-Ay-Bx)$ and $N(\xi)=N_{K/\mathbf{Q}}(\xi)$ with $K=Q(\omega)$, therefore (9) holds if and only if ξ is a unit in the field K. But, by Theorem 3, ξ is a unit in K if and only if $(x, y, z)=(G_n, G_{n+1}, G_{n+2})$ or $(x, y, z)=(-G_n, -G_{n+1}, -G_{n+2})$ for some integer n and so from $F(x, y, z)\geqq 0$

$$F(x, y, z) = |x| = |\pm G_n| = |G_n|$$

follows. This implies the statement since the function $F(x, y, z)$ takes all values $|G_n|$, namely $F(G_n, G_{n+1}, G_{n+2})=|G_n|$ for every integer n.

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