

On characterizations of the normal law in Hilbert space

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1. Introduction

KHATRI [5] gives some characterizations of multivariate normality through linear regressions and remarks that these results are valid on Hilbert space too, if the original conditions are replaced by conditions similar to those given by KUMAR and PATHAK [7]. But it is not necessary to change the conditions of Theorem 2 of [5]; it is true in Hilbert space in its original form.

The aim of this paper is to point out that most of the results and methods of [5] are "dimension free" (see [3] and [7]). Those lemmas of [5] that are valid only in finite dimensional spaces are replaced by lemmas which are valid in finite dimensional spaces as well as in Hilbert spaces.

2. Notations and preliminary lemmas

Let H, H_1, H_2 etc. denote real separable complete inner product spaces, that is separable Hilbert spaces or finite dimensional Euclidean spaces. For the sake of brevity these spaces are called Hilbert spaces. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. The adjoint of a linear bounded operator $A: H_1 \rightarrow H_2$ is denoted by A' . If we consider $h \in H$ as the following map from $\mathbf{R}$ into $H: h(x) = xh$ ($x \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers), then h' is the following map from H into $\mathbf{R}: h'(a) = \langle h, a \rangle$ ($a \in H$). In particular for fixed $h_1, h_2 \in H: h'_1 h'_2 = \langle h_1, h_2 \rangle$ and $h_1 h'_2$ is the following operator: $h_1 h'_2(a) = \langle h_2, a \rangle h_1 = \langle h_1, h'_2 a \rangle$ ($a \in H$).

If $H = \prod_{i=1}^n H_i$, where $H_1, \dots, H_n$ are Hilbert spaces, then H is a Hilbert space with the inner product:

$$\langle a, b \rangle = \sum_{i=1}^n \langle a_i, b_i \rangle, \quad a = (a_i)_{i=1}^n, \quad b = (b_i)_{i=1}^n \in H.$$

There is an isomorphism between the algebra of bounded linear operators of H and the algebra of matrices $(A_{ij})_{i,j=1}^n$, where $A_{ij}: H_j \rightarrow H_i$ is a bounded linear operator ($i, j = 1, \dots, n$). Through this paper I_{ii} denotes an appropriate identical operator.

Let ξ be a Hilbert space valued random variable (r.v.). $E\xi$, the expectation of ξ is defined by the Bochner integral of ξ . The covariance operator of ξ is $D^2\xi =$

$= E[(\xi - E\xi)(\xi - E\xi)']$. If $E|\xi|^2 < \infty$, then $D^2\xi$ exists. The conditional expectation $E(\xi|\cdot)$ is defined in Scalora [8]. ξ is said to have a normal distribution if for each constant vector h , the real-valued r.v. $\langle h, \xi \rangle$ has the normal distribution.

Lemma 1. Let ξ and η be H -valued r.v.'s, $E|\xi| < \infty$. $E(\xi|\eta)$ is constant if and only if

$$E\{\xi e^{i\langle t, \eta \rangle}\} = E\xi Ee^{i\langle t, \eta \rangle}$$

for each $t \in H$.

PROOF. See [4] and [2].

Lemma 2. (See [5].) Let ξ and η be independent Hilbert space valued r.v.'s, $E|\xi| < \infty$, $E|\eta| < \infty$.

(a) If $E(\xi + \eta|\eta)$ is constant, then η is degenerate.

(b) If $E(\eta|\xi + \eta)$ is constant and the characteristic functional of ξ nowhere vanishes, then η is degenerate.

PROOF. (a) is trivial; (b) follows from Lemma 1.

Lemma 3. (See [6].) Let ξ and (η, ζ) be independent Hilbert space valued r.v.'s. Assume that the characteristic functional of ξ nowhere vanishes. If $\xi + \eta$ and ζ are independent, then η and ζ are independent.

PROOF. The proof is based on elementary properties of the characteristic functional.

Lemma 4. (See [5].) Let $H = \prod_{i=1}^n H_i$ and let $A = A_{(n)} = (A_{ij})_{i,j=1}^n$ be a bounded linear operator of H . Assume that $A_{(r)} = (A_{ij})_{i,j=1}^r$ is an invertible linear operator for $r = 1, 2, \dots, n-1$. Then there exists a decomposition $A = TS$, where $T = (T_{ij})_{i,j=1}^n$ is lower triangular with

$$T_{ii} = A_{ii} - (A_{i1}, \dots, A_{i,i-1})[A_{(i-1)}]^{-1}(A'_{1i}, \dots, A'_{i-1,i})' = A_{ii} - \sum_{k=1}^{i-1} T_{ik}S_{ki}$$

($i = 2, 3, \dots, n$). T_{ii} is invertible for $i = 1, 2, \dots, n-1$. (If $A_{(n)}$ is invertible, then T_{nn} is also invertible.) $S = (S_{ij})_{i,j=1}^n$ is upper triangular and $S_{ii}: H_i \rightarrow H_i$ is identical ($i = 1, \dots, n$).

PROOF. For $k = 2$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} I_{11} & A_{11}^{-1}A_{12} \\ 0 & I_{22} \end{pmatrix}$$

is the required decomposition. Then one can proceed by induction.

Lemma 5. (See [5].) Let ζ and (ξ, η_1, η_2) be independent Hilbert space valued r.v.'s and let ζ have normal distribution. Let $\zeta_1 = \zeta + A_1\xi + \eta_1$, $E|\zeta_1| < \infty$. Assume that $E(\zeta_1|\zeta - B_1\xi, \eta_1, \eta_2) = 0$, where A_1 and B_1 are bounded linear operators satisfying the relation $AB_1 = BA_1$ for some operators A and B . Then $(\zeta, B\zeta_1)$ is normally distributed and $(\zeta, B\zeta_1)$ and (η_1, η_2) are independent.

PROOF. Let $\mu = A_1 \zeta + \eta_1$. From the conditions of the lemma

$$E(\zeta + \mu | A\zeta - B\mu, \eta_1, \eta_2) = 0.$$

By Lemma 1

$$E\{(\zeta + \mu) \exp[i\langle A\zeta - B\mu, t \rangle + i\langle \eta_1, t_1 \rangle + i\langle \eta_2, t_2 \rangle]\} = 0.$$

Let $f_1(s)$ and $f_2(s, t_1, t_2)$ denote the characteristic functional of ζ and (μ, η_1, η_2) respectively. Then

$$\left. \frac{df_1(s)}{ds} \right|_{s=A't} f_2(-B't, t_1, t_2) + \left. \frac{\partial f_2(s, t_1, t_2)}{\partial s} \right|_{s=-B't} f_1(A't) = 0,$$

where $\frac{df_1(s)}{ds}$ and $\frac{\partial f_2(s, t_1, t_2)}{\partial s}$ denote the Fréchet derivative and the Fréchet partial derivative (cf. [1]) resp. Since ζ is normally distributed we have (see [9])

$$f_1(s) = \exp\left\{i\langle m, s \rangle - \frac{1}{2}\langle \Sigma s, s \rangle\right\}.$$

Therefore from the above equation

$$B(im - \Sigma A't) f_2(-B't, t_1, t_2) = \frac{\partial f_2(-B't, t_1, t_2)}{\partial t}.$$

This shows that

$$f_2(-B't, t_1, t_2) = \exp\left\{i\langle Bm, t \rangle - \frac{1}{2}\langle B\Sigma A't, t \rangle\right\} g(t_1, t_2).$$

Hence $B\mu$ is normally distributed and $B\mu$ and (η_1, η_2) are independent. This proves the lemma.

Lemma 6. (See [5].) Let ζ and (μ, η) be independent Hilbert space valued r.v.'s. Let

$$\zeta_0 = A\zeta + (A - I)\mu,$$

where A is an arbitrary bounded linear operator and I is the identical operator. Assume that $E|\zeta_0|^2 < \infty$ and

$$(1) \quad E(\zeta_0 | \zeta + \mu, \eta) = 0;$$

$$(2) \quad E(\zeta_0 \zeta'_0 | \zeta + \mu, \eta) = \Sigma,$$

where Σ is a constant operator.

Then $A\zeta$ and $(A - I)\mu$ are normally distributed. If A is invertible, then ζ , $(A - I)\mu$ and η are independent.

PROOF. Let $\varphi_1(t)$ and $\varphi_2(t, t_1)$ be the characteristic functional of ζ and (μ, η) respectively. With the help of Lemma 1 from (1) and (2) one can get

$$(3) \quad A \frac{d\varphi_1(t)}{dt} \varphi_2(t, t_1) + (A - I) \frac{\partial \varphi_2(t, t_1)}{\partial t} \varphi_1(t) = 0$$

and

$$(4) \quad A \frac{d^2 \varphi_1(t)}{dt^2} A' \varphi_2(t, t_1) + A \frac{d\varphi_1(t)}{dt} \left[\frac{\partial \varphi_2(t, t_1)}{\partial t} \right]' (A-I)' + \\ + (A-I) \frac{\partial \varphi_2(t, t_1)}{\partial t} \left[\frac{d\varphi_1(t)}{dt} \right]' A' + (A-I) \frac{\partial^2 \varphi_2(t, t_1)}{\partial t^2} (A-I)' \varphi_1(t) = \\ = -\Sigma \varphi_1(t) \varphi_2(t, t_1).$$

From (3) and (4) we obtain for t and t_1 in some neighborhood of the origin:

$$(5) \quad A \frac{d^2 \varphi_1(t)}{dt^2} \frac{1}{\varphi_1(t)} - A \frac{d\varphi_1(t)}{dt} \left[\frac{d\varphi_1(t)}{dt} \right]' \frac{1}{\varphi_1^2(t)} = -\Sigma$$

and

$$(A-I) \frac{\partial^2 \varphi_2(t, t_1)}{\partial t^2} \frac{1}{\varphi_2(t, t_1)} - (A-I) \frac{\partial \varphi_2(t, t_1)}{\partial t} \left[\frac{\partial \varphi_2(t, t_1)}{\partial t} \right]' \frac{1}{\varphi_2^2(t, t_1)} = \Sigma.$$

From these equations we have

$$\frac{d^2 \ln [\varphi_1(A't)]}{dt^2} = -\Sigma A'$$

and

$$\frac{\partial^2 \ln \{\varphi_2[(I-A)'t, t_1]\}}{\partial^2 t} = -\Sigma(I-A)'.$$

By the theorem of Marcinkiewicz (see Lemma 2.4.3 of [4]) $A'\zeta$ and $(I-A)\mu$ are normally distributed.

If A is invertible, then from (5) we have

$$\varphi_1(t) = \exp \left\{ i \langle m, t \rangle - \frac{1}{2} \langle A^{-1} \Sigma t, t \rangle \right\}.$$

In this case (3) can be rewritten in the form

$$(iAm - \Sigma t) \varphi_2(t, t_1) + (A-I) \frac{\partial \varphi_2(t, t_1)}{\partial t} = 0.$$

Let $t = (I-A)'s$, then

$$[iAm - \Sigma(I-A)'s] \bar{\varphi}_2(s, t_1) - \frac{\partial \bar{\varphi}_2(s, t_1)}{\partial s} = 0,$$

where $\bar{\varphi}_2(s, t_1) = \varphi_2[(I-A)'s, t_1]$ is the characteristic functional of $[(I-A)\mu, \eta]$. Its general solution is

$$\bar{\varphi}_2(s, t_1) = \exp \left\{ i \langle Am, s \rangle - \frac{1}{2} \langle \Sigma(I-A)'s, s \rangle \right\} \psi(t_1),$$

where ψ is a function of t_1 only. This shows us that if A is invertible, then $(I-A)\mu$ and η are independent. Thus the lemma is proved.

In Theorem 2 we shall need the following facts on conditional distributions. Let H_1 and H_2 be Hilbert spaces and let (ξ, η) be a $H_1 \times H_2$ -valued r.v. Assume that

(ξ, η) has canonical form. Denote by $Q_{\xi, \eta}$, Q_ξ and Q_η the distribution of (ξ, η) , ξ and η respectively. Let $P_{\xi|\eta=y}(C; y)$ denote a regular conditional distribution of ξ given $\eta=y$, where $y \in H_2$ and $C \in \mathcal{B}(H_1)$, the Borel σ -algebra of H_1 . If

$$P_{\xi|\eta=y}(C + E(\xi|\eta = y); y) = P(C), \quad C \in \mathcal{B}(H_1), \quad y \in H_2,$$

where the distribution $P(\cdot)$ does not depend on y , we shall say, that the conditional distribution of ξ given η depends on η only through the conditional expectation.

Lemma 7. *If the above conditions are satisfied, then $\xi - E(\xi|\eta)$ and η are independent.*

PROOF. Let $C \in \mathcal{B}(H_1)$, $D \in \mathcal{B}(H_2)$.

$$\begin{aligned} Q_{\xi - E(\xi|\eta), \eta}(C \times D) &= P([\xi - E(\xi|\eta)] \in C, \eta \in D) \\ &= P(\xi \in C + E(\xi|\eta), \eta \in D) \\ &= \int_D P_{\xi|\eta=y}(C + E(\xi|\eta = y); y) Q_\eta(dy) \\ &= \int_D P(C) Q_\eta(dy). \end{aligned}$$

Hence the conditional distribution of $\xi - E(\xi|\eta)$ given $\eta=y$ does not depend on y , that is $\xi - E(\xi|\eta)$ and η are independent.

3. Characterization theorems

Proposition 1. (See [5].) *Let $\xi_1, \xi_2, \eta_1, \eta_2$ be Hilbert space valued r.v.'s. Let*

$$\zeta_1 = \xi_1 + A_{12}\xi_2 + \eta_1$$

and

$$\zeta_2 = A_{21}\xi_1 + \xi_2 + \eta_2.$$

Let $E|\zeta_2|^2 < \infty$. Assume that $A = A_{12}A_{21}$ is invertible and the range of $B = I - A$ is closed. If

- (i) ζ_1 and (ξ_2, η_1, η_2) are independent;
- (ii) $E(\zeta_2|\xi_1, \eta_1, \eta_2) = 0$;

and

- (iii) $E(\zeta_2\zeta_2'|\xi_1, \eta_1, \eta_2) = \Sigma$,

where Σ is constant, then $(A\zeta_1, A_{12}\zeta_2)$ is normally distributed and $(\zeta_1, A_{12}\zeta_2)$ and (η_1, η_2) are independent.

PROOF. Let $\xi_{(2)} = A_{12}\xi_2$ and $\eta_{(2)} = \eta_2 - A_{21}\eta_1$, then

$$\zeta_0 = A_{12}\zeta_2 = A\zeta_1 + (I - A)\xi_{(2)} + A_{12}\eta_{(2)}.$$

From (ii) and (iii) we obtain

$$(6) \quad E(\zeta_0 | \zeta_1 - \xi_{(2)}, \eta_{(2)}, \eta_1) = 0,$$

$$(7) \quad E(\zeta_0 \zeta'_0 | \zeta_1 - \xi_{(2)}, \eta_{(2)}, \eta_1) = A_{12} \Sigma A'_{12} = \Sigma_1.$$

There exists a linear bounded operator B^- such that $BB^-B=B$. From (6) we then find

$$E\{(I-BB^-)A\zeta_1 + (I-BB^-)A_{12}\eta_{(2)} | \eta_{(2)}\} = 0.$$

By Lemma 2 $(I-BB^-)A_{12}\eta_{(2)}$ is degenerate.

We can assume that $EA_{12}\eta_{(2)}=0$, $E(I-A)\xi_{(2)}=0$ and $EA\zeta_1=0$. It follows that $(I-BB^-)A_{12}\eta_{(2)}=0$. Let $\mu=\xi_{(2)}+B^-A_{12}\eta_{(2)}$, then

$$(I-A)\mu = (I-A)\xi_{(2)} + A_{12}\eta_{(2)}.$$

Hence $\zeta_0=A\zeta_1+(I-A)\mu$. Then (i), (6) and (7) give

$$(8) \quad \zeta_1 \text{ and } (\mu, \eta) \text{ are independent;}$$

$$(9) \quad E(\zeta_0 | \zeta_1 - \mu, \eta) = 0;$$

$$(10) \quad E(\zeta_0 \zeta'_0 | \zeta_1 - \mu, \eta) = \Sigma_1,$$

where $\eta=(\eta_1, \eta_2)$. The required result follows from Lemma 6.

Proposition 2. (See [6].) *Let $\xi_1, \xi_2, \eta_1, \eta_2$ be Hilbert space valued r.v.'s. Let*

$$\zeta_1 = \xi_1 + A_{12}\xi_2 + \eta_1$$

and

$$\zeta_2 = A_{21}\xi_1 + \xi_2 + \eta_2.$$

If ζ_1 and (ξ_2, η_1, η_2) are independent and ζ_2 and (ξ_1, η_1, η_2) are independent, then $(A_{21}\xi_1, A_{12}\xi_2)$ is normally distributed and (ζ_1, ζ_2) and (η_1, η_2) are independent.

PROOF. Let $T_{22}=I-A_{21}A_{12}$ and $\eta=\eta_2-A_{21}\eta_1$. Then

$$\xi_1 = \zeta_1 - A_{12}\xi_2 - \eta_1$$

and

$$(11) \quad \zeta_2 = A_{21}\zeta_1 + T_{22}\xi_2 + \eta.$$

From the assumptions of the proposition $A_{21}\zeta_1 + T_{22}\xi_2 + \eta$ and $(\zeta_1 - A_{12}\xi_2, \eta)$ are independent. Hence

$$(12) \quad \begin{aligned} & \varphi(A'_{21}t_1 + A'_{21}t_2)\psi(T'_{22}t_1 - A'_{12}A'_{21}t_2, t_1 + t_3) = \\ & = \varphi(A'_{21}t_1)\psi(T'_{22}t_1, t_1)\varphi(A'_{21}t_2)\psi(-A'_{12}A'_{21}t_2, t_3), \end{aligned}$$

where φ and ψ denotes the characteristic functional of ζ_1 and (ξ_2, η) respectively. Let $\bar{\varphi}$ and $\bar{\psi}$ be the characteristic functional of $(\zeta_1, 0, 0)$ and $(\xi_2, \eta, 0)$ respectively.

Since $\bar{\varphi}$ and $\bar{\psi}$ do not vanish in a neighborhood of the origin, we can take their logarithm in this neighborhood. Then from (12) we have

$$f(B_1t) + g(B_2t) = f(B_3t + g(B_4t) + f(B_5t) + g(B_6t))$$

in a neighborhood of 0, where $t' = (t'_1, t'_2, t'_3)$, $f(t) = \log \bar{\varphi}(t)$, $g(t) = \log \bar{\psi}(t)$ and

$$B_1 = \begin{pmatrix} A'_{21} & A'_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} T'_{22} & -A'_{12}A'_{21} & 0 \\ I & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} A'_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} T'_{22} & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & A'_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & -A'_{12}A'_{21} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the conditions of Lemma 2.3 of [7] are satisfied. It follows that $f(B_1 t)$ is a polynomial in a neighborhood of $t=0$. Marcinkiewicz's theorem implies that $A_{21}\zeta_1$ is normal. By similar arguments we get that $A_{12}\zeta_2$ has also a normal distribution. By equation (11)

$$A_{12}\zeta_2 = A_{12}(A_{21}\zeta_1) + A_{12}(T_{22}\zeta_2 + \eta).$$

Cramér's theorem implies that $A_{12}(T_{22}\zeta_2 + \eta)$ is normal. Therefore it follows that $(A_{21}\zeta_1, A_{12}\zeta_2)$ is normally distributed.

Finally Lemma 3 shows that (ζ_1, ζ_2) and (η_1, η_2) are independent.

Theorem 1. (See Theorem 2 of [5].) *Let $X_1, \dots, X_n, U_1, \dots, U_n$ be Hilbert space valued random variables and let*

$$Z_j = \sum_{i=1}^n A_{ji} X_i + U_j,$$

where $A_{jj} = I_{jj}$ ($j=1, 2, \dots, n$). Suppose that the following conditions are satisfied:

- (i) Z_1 and $(X_2, X_3, \dots, X_n, U_1, \dots, U_n)$ are independent;
- (ii) $E(Z_j | X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n, U_1, \dots, U_n) = 0$ for $j=2, 3, \dots, n$;
- (iii) $E(Z_2 Z_2' | X_1, X_3, \dots, X_n, U_1, \dots, U_n) = \Sigma$,

where Σ is constant.

Let $A_{(i)} = (A_{ji})_{j=1}^i$ be invertible $i=1, 2, \dots, n$. Assume that A_{12}, A_{21} and $I - T_{ii}$ ($i=3, \dots, n$) are invertible, where T_{ii} ($i=1, \dots, n$) are defined as in Lemma 4.

Then $(Z_1, \dots, Z_n)$ is normally distributed and $(Z_1, \dots, Z_n)$ and $(U_1, \dots, U_n)$ are independent.

PROOF. According to the decomposition given in Lemma 4 we can write $A = A_{(n)} = TS$. Let

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}, \quad W = SX + T^{-1}U,$$

then $Z = TW$. With these notations $Z_j = \sum_{k=1}^j T_{jk} W_k$ and

$$\begin{aligned} & \sigma\{X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n, U\} \supset \\ & \supset \sigma\{W_1 - S_{1j} W_j, \dots, W_{j-1} - S_{j-1,j} W_j, W_{j+1}, \dots, W_n, U\}, \end{aligned}$$

where σ denotes the generated σ -algebra. Then (i), (ii) and (iii) give:

- (i)' Z_1 and $(W_2, \dots, W_n, U)$ are independent;
 (ii)' $E(Z_j | W_1 - S_{1j}W_j, \dots, W_{j-1} - S_{j-1,j}W_j, W_{j+1}, \dots, W_n, U) = 0$ for

$j = 2, 3, \dots, n$;

- (iii)' $E(Z_2 Z_2' | W_1 - S_{12}W_2, W_3, \dots, W_n, U) = \Sigma$.

Now we can apply Proposition 1 with $\xi_1 = W_1 - S_{21}W_2$, $\xi_2 = W_2$, $\eta_1 = 0$, $\eta_2 = 0$. It follows that $(\xi_1, \xi_2) = (Z_1, Z_2)$ has a normal distribution.

We proceed by induction. Assume that

- (13) $(Z_1, \dots, Z_{l-1})$ is normally distributed for $1 < l \leq j$;

and

- (14) $(Z_1, \dots, Z_{l-1})$ and $(W_l, \dots, W_n, U)$ are independent for $1 < l \leq j$, where $1 < j \leq n$. (It follows that $W_1, \dots, W_{j-1}$ are independent normal r.v.'s. and $(W_1, \dots, W_{j-1})$ and $(W_j, \dots, W_n, U)$ are independent.)

Let

$$\begin{aligned}\zeta &= (T_{j1}, \dots, T_{j,j-1})[T_{(j-1)}]^{-1}(Z_1', \dots, Z_{j-1}')', \\ \xi &= W_j, \quad \eta_1 = 0, \quad \eta_2 = (W_{j+1}, \dots, W_n, U),\end{aligned}$$

where $T_{(j-1)} = (T_{il})_{i,l=1}^{j-1}$. Then from (13) and (14)

- (15) ζ and (ξ, η_1, η_2) are independent;

- (16) ζ is normally distributed;

and from (ii)

- (17) $E(\zeta_1 | \zeta - (I - T_{jj})\zeta, \eta_1, \eta_2) = 0$,

where $\zeta_1 = \zeta + T_{jj}\zeta = \sum_{k=1}^j T_{jk}W_k = Z_j$. Then Lemma 5 gives that Z_j is normally

distributed. Hence W_j is normally distributed and $W_1, \dots, W_j$ are independent normal r.v.'s, whence $(Z_1, \dots, Z_j)$ is normal. It follows from Lemma 5 that $(Z_1, \dots, Z_j)$ and $(W_{j+1}, \dots, W_n, U)$ are independent. This completes the proof.

Corollary 1. In Theorem 1, condition (iii) can be replaced by the following condition:

" Z_2 and $(X_1, X_3, \dots, X_n, U_1, \dots, U_n)$ are independent".

PROOF. By Proposition 2 (Z_1, Z_2) is normally distributed. Then one can proceed as in the proof of Theorem 1.

Theorem 2. (See Note 5 of [5].) Let $X_1, \dots, X_n$ be Hilbert space valued r.v.'s. Assume that

- (a) the regression of X_i on the rest of the variables is linear, that is

$$\begin{aligned}E(X_i | X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) &= \\ &= -[A_{i1}X_1 + \dots + A_{i,i-1}X_{i-1} + A_{i,i+1}X_{i+1} + \dots + A_{in}X_n] \quad (i = 1, \dots, n);\end{aligned}$$

(b) the conditional distribution of X_1 depends on the rest of the variables only through the conditional expectation;

(c) $D^2(X_2|X_1, X_3, \dots, X_n) = \Sigma$, where Σ is constant.

Let $A = (A_{ij})_{i,j=1}^n$ with $A_{ii} = I_{ii}$ ($i = 1, \dots, n$). Suppose that A satisfies the conditions given in Theorem 1.

Then $(X_1, \dots, X_n)$ is normal.

PROOF. Let $X' = (X'_1, \dots, X'_n)$ and let $Z = AX$. (a) and (c) imply that the conditions (ii) and (iii) of Theorem 1 are satisfied. Condition (i) of Theorem 1 follows from (a) and (b) with the help of Lemma 7. Therefore X is normal.

Corollary 2. In Theorem 2, condition (c) can be replaced by the condition

(c)' the conditional distribution of X_2 depends on the rest of the variables only through its conditional expectation.

PROOF. It is an easy consequence of Corollary 1.

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