

On inequalities concerning the permanents of generalized doubly stochastic matrices

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1. Introduction

a) Let R_n denote the n -dimensional real vector space with column vectors as its elements. Let M denote the set of $n \times n$ matrices with real elements. Let A^* denote the transpose of $A \in M$. $E \in M$ is the unit matrix.

Let $K \subset M$ denote the set of matrices, where all row and column sums are 1. Let $H \subset K$ be the set of matrices with non-negative elements, i.e. the set of so-called doubly stochastic matrices. Let $S_0 \in H$ be the matrix where all the entries are $1/n$.

The elements of the set K are said to be generalized doubly stochastic matrices.

If $A = (a_{jk}) \in M$ then the permanent of A , denoted by $\text{Per } A$, is defined as follows:

$$\text{Per } A = \sum_{(i_1, \dots, i_n)} a_{1i_1} \dots a_{ni_n},$$

where $(i_1, \dots, i_n)$ runs over the full symmetric group.

Let Γ be the set of vectors $(\beta_1, \dots, \beta_n)$, where the components are non-negative integers satisfying the conditions

$$0 \leq \beta_k \leq n \quad (k = 1, \dots, n), \quad \beta_1 + \dots + \beta_n = n.$$

Let $C_{\beta_1 \dots \beta_n}(A) \in M$ denote the matrix, which consists of certain elements of $A \in M$. Namely the k -th column of A appears β_k -times ($k = 1, \dots, n$) in $C_{\beta_1 \dots \beta_n}(A)$, where $(\beta_1, \dots, \beta_n)$ runs over Γ .

Let $A = U \wedge V^*$ be the polar representation of $A \in M$, where $U \in M$, $V \in M$, $UU^* = VV^* = E$, and $\wedge$ is a diagonal matrix with diagonal elements $\lambda_j \geq 0$ ($j = 1, \dots, n$). Using the Cauchy—Binet expansion theorem ([3], 579), we get

$$\text{Per } A = \sum_{(\beta_1, \dots, \beta_n) \in \Gamma} \frac{\lambda_1^{\beta_1} \dots \lambda_n^{\beta_n}}{\beta_1! \dots \beta_n!} \text{Per } C_{\beta_1 \dots \beta_n}(U) \text{Per } C_{\beta_1 \dots \beta_n}(V).$$

b) The following Conjecture is due to VANDER WAERDEN ([3], 586, Conjecture 1): If $A \in H$, then $\text{Per } A \geq n!/n^n$, with equality if and only if $A = S_0$. This conjecture was proved by G. P. JEGORITSEV in his paper [2].

In his paper [1] the author proved the following two theorems of Van der Waerden type, which are not a consequence of the Van der Waerden—Jegoritsev theorem because $H \subset K$.

If $(AA^*)^{1/2}$ denotes the only non-negative definite square root of AA^* , then we have the following theorem:

Theorem 1.1. *If $A \in K$ and if $x \in R_1, y \in R_1, x+y=1$, then*

$$x^2 \text{Per} (AA^*)^{1/2} + y^2 \text{Per} (A^*A)^{1/2} + 2xy \text{Per} A \cong n!/n^n$$

with equality if and only if $A = S_0$.

Theorem 1.2. *If $A \in K$ with polar representation $A = U \wedge V^*$, and if*

$$\text{Per } C_{\beta_1 \dots \beta_n}(U) \text{Per } C_{\beta_1 \dots \beta_n}(V) \cong 0$$

$$(\beta_1, \dots, \beta_n) \in \Gamma,$$

then $\text{Per} A \cong n!/n^n$ with equality if and only if $A = S_0$.

In this paper we give extensions of these two theorems.

2. The extension of the first Theorem

Let $A = U \wedge V^*$ be the polar representation of the matrix $A \in K$. Let $\lambda_k \cong 0$ ($k=1, \dots, n$) be the elements of the diagonal matrix $\wedge$. Denote by u_k and v_k ($k=1, \dots, n$) the k -th column of the matrix U and V , respectively.

As it is known $A \in K$ if and only if one of λ_k ($k=1, \dots, n$) is equal to 1 and simultaneously all elements of the corresponding column of U and V are equal to $1/\sqrt{n}$. In the following, without loss of the generality, let us suppose that $\lambda_1=1$ and thus all elements of u_1 and v_1 are $1/\sqrt{n}$.

Let $1 \cong r + \alpha \cong n$, where α and r are non-negative integers. Let $i_1, \dots, i_r$ a combination of order r of the elements $\alpha+1, \dots, n$, without repetition and without permutation, i.e.

$$\alpha+1 \cong i_1 < \dots < i_r \cong n.$$

Let

$$U_{i_1 \dots i_r}^{(\alpha)} = (u_1 \dots u_\alpha u_{i_1} \dots u_{i_r}),$$

$$V_{i_1 \dots i_r}^{(\alpha)} = (v_1 \dots v_\alpha v_{i_1} \dots v_{i_r}),$$

$$A_{i_1 \dots i_r}^{(\alpha)} = \begin{pmatrix} \lambda_1 & & & & (0) \\ & \ddots & & & \\ & & \lambda_\alpha & & \\ & & & \lambda_{i_1} & \\ (0) & & & & \ddots & \lambda_{i_r} \end{pmatrix}.$$

Then obviously

$$A_{i_1 \dots i_r}^{(\alpha)} = U_{i_1 \dots i_r}^{(\alpha)} A_{i_1 \dots i_r}^{(\alpha)} V_{i_1 \dots i_r}^{(\alpha)*} \in M.$$

Let

$$G_r^{(x)}(x, y) = \frac{1}{\binom{n-\alpha}{r}} \sum_{\alpha+1 \leq i_1 < \dots < i_r \leq n} \{x^2 \text{Per}(A_{i_1 \dots i_r}^{(x)} A_{i_1 \dots i_r}^{(x)*})^{1/2} + \\ + y^2 \text{Per}(A_{i_1 \dots i_r}^{(x)*} A_{i_1 \dots i_r}^{(x)})^{1/2} + 2xy \text{Per} A_{i_1 \dots i_r}^{(x)}\},$$

where $x \in R_1, y \in R_1$.

Theorem 2.1. I. If $1 \leq \alpha \leq n$ then

$$(2.1) \quad A_{i_1 \dots i_r}^{(x)} \in K.$$

II. If $x \in R_1, y \in R_1, x+y=1$ then

$$G_{r+1}^{(x)}(x, y) \cong G_r^{(x)}(x, y)$$

$$(r = 0, 1, \dots, n-\alpha-1; \alpha = 0, 1, \dots, n-1; 1 \leq \alpha+r < n)$$

with equality if and only if $\lambda_{\alpha+1} = \dots = \lambda_n = 0$.

PROOF. ad I. As a consequence of the condition we get that the first element of the diagonal matrix $A_{i_1 \dots i_r}^{(x)}$ is equal to 1, moreover all elements of the first column of the matrices $U_{i_1 \dots i_r}^{(x)}$ and $V_{i_1 \dots i_r}^{(x)}$ are $1/\sqrt{n}$, i.e. (2.1) turned out to be right ([1], 108).

ad II. Taking into account that by the Cauchy—Binet expansion theorem

$$x^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(x)} A_{i_1 \dots i_{r+1}}^{(x)*})^{1/2} + y^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(x)*} A_{i_1 \dots i_{r+1}}^{(x)})^{1/2} + 2xy \text{Per} A_{i_1 \dots i_{r+1}}^{(x)} =$$

$$= \sum_{\beta_1 + \dots + \beta_{\alpha+r} = n} \frac{\lambda_1^{\beta_1} \dots \lambda_{\alpha}^{\beta_{\alpha}} \lambda_{i_r+1}^{\beta_{\alpha+1}} \dots \lambda_{i_r+r}^{\beta_{\alpha+r}}}{\beta_1! \dots \beta_{\alpha}! \beta_{\alpha+1}! \dots \beta_{\alpha+r}!} \times \\ \times [x \text{Per} C_{\beta_1 \dots \beta_{\alpha+r}}(U_{i_1 \dots i_r}^{(x)}) + y \text{Per} C_{\beta_1 \dots \beta_{\alpha+r}}(V_{i_1 \dots i_r}^{(x)})]^2 + \\ + \sum_{\substack{\beta_1 + \dots + \beta_{\alpha+r+1} = n \\ \beta_{\alpha+r+1} \geq 1}} \frac{\lambda_1^{\beta_1} \dots \lambda_{\alpha}^{\beta_{\alpha}} \lambda_{i_1}^{\beta_{\alpha+1}} \dots \lambda_{i_r+1}^{\beta_{\alpha+r+1}}}{\beta_1! \dots \beta_{\alpha}! \beta_{\alpha+1}! \dots \beta_{\alpha+r+1}!} \times \\ \times [x \text{Per} C_{\beta_1 \dots \beta_{\alpha+r+1}}(U_{i_1 \dots i_{r+1}}^{(x)}) + y \text{Per} C_{\beta_1 \dots \beta_{\alpha+r+1}}(V_{i_1 \dots i_{r+1}}^{(x)})]^2,$$

we obtain

$$(2.2) \quad x^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(x)} A_{i_1 \dots i_{r+1}}^{(x)*})^{1/2} + y^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(x)*} A_{i_1 \dots i_{r+1}}^{(x)})^{1/2} + \\ + 2xy \text{Per} A_{i_1 \dots i_{r+1}}^{(x)} \cong x^2 \text{Per}(A_{i_1 \dots i_r}^{(x)} A_{i_1 \dots i_r}^{(x)*})^{1/2} + \\ + y^2 \text{Per}(A_{i_1 \dots i_r}^{(x)*} A_{i_1 \dots i_r}^{(x)})^{1/2} + 2xy \text{Per} A_{i_1 \dots i_r}^{(x)}$$

with equality if and only if $\lambda_{i_r+1} = 0$. Namely if $\lambda_{i_r+1} > 0$ and since $\lambda_1 = 1$, we get equality if and only if

$$(2.3) \quad x \text{Per} C_{\beta_1=i, \beta_{i_r+1}=n-i}(U_{i_1 \dots i_{r+1}}^{(x)}) + y \text{Per} C_{\beta_1=i, \beta_{i_r+1}=n-i}(V_{i_1 \dots i_{r+1}}^{(x)}) = 0 \\ (i = 0, \dots, n-1).$$

Let $r_1, \dots, r_n$ and $s_1, \dots, s_n$ the components of the vectors $u_{i_{r+1}}$ and $v_{i_{r+1}}$, respectively. Using the notation

$$G_i(x_1, \dots, x_n) = \sum_{1 \leq \alpha_1 < \dots < \alpha_i \leq n} x_{\alpha_1} \dots x_{\alpha_i}$$

$$(i = 1, \dots, n),$$

we get from (2,3) the system of equations

$$(2.4) \quad xG_i(r_1, \dots, r_n) + yG_i(s_1, \dots, s_n) = 0$$

$$(i = 1, \dots, n).$$

Since $x+y=1$ we may suppose, without loss of the generality, that $x \neq 0$. Thus from (2,4)

$$(2.5) \quad G_i(r_1, \dots, r_n) = cG_i(s_1, \dots, s_n)$$

$$(i = 1, \dots, n),$$

where $c = -\frac{x}{y}$. Let

$$f(x) = (x-r_1)\dots(x-r_n), \quad g(x) = c(x-s_1)\dots(x-s_n).$$

In consequence of (2,5) $f(x) \equiv g(x)$. Thus $c=1$, i.e. $x+y=0$, what is impossible because $x+y=1$. Therefore $\lambda_{i_{r+1}}=0$.

If i_{r+1} runs over all the numbers $\alpha+1, \dots, n$ different from $i_1, \dots, i_r$, then on the basis (2,2) we get that

$$(2.6) \quad \frac{1}{n-\alpha-r} \sum_{(i_{r+1})} x^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(\alpha)} A_{i_1 \dots i_{r+1}}^{(\alpha)*})^{1/2} +$$

$$+ y^2 \text{Per}(A_{i_1 \dots i_{r+1}}^{(\alpha)*} A_{i_1 \dots i_{r+1}}^{(\alpha)})^{1/2} + 2xy \text{Per} A_{i_1 \dots i_{r+1}}^{(\alpha)} \cong$$

$$\cong x^2 \text{Per}(A_{i_1 \dots i_r}^{(\alpha)} A_{i_1 \dots i_r}^{(\alpha)*})^{1/2} + y^2 \text{Per}(A_{i_1 \dots i_r}^{(\alpha)*} A_{i_1 \dots i_r}^{(\alpha)})^{1/2} + 2xy \text{Per} A_{i_1 \dots i_r}^{(\alpha)}$$

with equality if and only if the numbers $\lambda_1, \dots, \lambda_n$ different from $\lambda_1, \dots, \lambda_\alpha, \lambda_{i_1}, \dots, \lambda_{i_r}$, are zero.

Using formula (2,6) for all combinations $i_1, \dots, i_r$ of order r of the elements $\alpha+1, \dots, n$, without repetition and without permutation, it can be shown that the expression of the square brackets on the left hand side of (2,6) appears $r+1$ times among these inequalities. Calculating the arithmetic mean of these inequalities, i. e. dividing both side of the sum of these inequalities by $\binom{n-\alpha}{r}$, after all we divide the left hand side and the right hand side of the sum of these inequalities by $\frac{n-\alpha-r}{r+1} \binom{n-\alpha}{r} = \binom{n-\alpha}{r+1}$ and by $\binom{n-\alpha}{r}$, respectively. Thus we obtain the inequality of our Theorem 2.1. with equality if and only if $\lambda_{\alpha+1} = \dots = \lambda_n = 0$.

Corollary 2.1. *If $A \in K$, $x \in R_1$, $y \in R_1$, $x+y=1$, then*

$$G_{r+1}^{(1)}(x, y) \cong G_r^{(1)}(x, y) \cong \frac{n!}{n^n} \quad (r = 0, 1, \dots, n-2)$$

with equality if and only if $A = S_0$.

PROOF. In this case all components of the vectors U_1 and v_1 are $1/\sqrt{n}$, thus $G_0^{(1)}(x, y) = n!/n^n$. Equality if and only if $\lambda_2 = \dots = \lambda_n = 0$.

We get Theorem 1.1 from Corollary 2.1 in the case $r+1 = n-1$.

Corollary 2.2. *If $A = U \wedge V^*$ is the polar representation of $A \in K$ and if $x \in R_1$, $y \in R_1$, $x+y=1$ then*

I. $A_{i_1 \dots i_r}^{(0)} \in K$ if and only if $i_1 = 1$.

II. $G_{r+1}^{(0)}(x, y) > G_r^{(0)}(x, y)$ ($r = 1, \dots, n-1$).

PROOF. $A_{i_1 \dots i_r}^{(0)} \in K$ if and only if one of $\lambda_1, \dots, \lambda_n$ is equal to 1 and the corresponding column-elements of U and V are $1/\sqrt{n}$. U and V are orthogonal matrices, therefore these conditions are satisfied only if $j=1$.

In the case II. equality holds if and only if $\lambda_1 = \dots = \lambda_n = 0$ contradicting to $\lambda_1 = 1$.

3. The extension of the second Theorem

Let

$$G_r^{(\alpha)}(A) = \frac{1}{\binom{n-\alpha}{r}} \sum_{\alpha+1 \leq i_1 < \dots < i_r \leq n} \text{Per } A_{i_1 \dots i_r}^{(\alpha)}$$

$$(r = 0, 1, \dots, n-\alpha-1; \alpha = 0, 1, \dots, n-1; r+\alpha \geq 1).$$

Theorem 3.1. *If the matrix $A \in K$ has the polar representation $A = U \wedge V^*$ and if*

$$(3.1) \quad \text{Per } C_{\beta_1 \dots \beta_n}(U) \text{Per } C_{\beta_1 \dots \beta_n}(V) \geq 0.$$

$$(\beta_1, \dots, \beta_n) \in \Gamma,$$

then

$$(3.2) \quad G_{r+1}^{(\alpha)}(A) \geq G_r^{(\alpha)}(A)$$

$$(\alpha = 0, 1, \dots, n-1; r = 0, 1, \dots, n-\alpha-1, r+\alpha \geq 1)$$

with equality if and only if $\lambda_{\alpha+1} = \dots = \lambda_n = 0$.

PROOF. Applying again the Cauchy—Binet expansion theorem, we get

$$\begin{aligned} \text{Per } A_{i_1 \dots i_{r+1}}^{(\alpha)} &= \sum_{\beta_1 + \dots + \beta_{\alpha+r} = n} \frac{\lambda_1^{\beta_1} \dots \lambda_{\alpha}^{\beta_{\alpha}} \lambda_{i_1}^{\beta_{\alpha+1}} \dots \lambda_{i_r}^{\beta_{\alpha+r}}}{\beta_1! \dots \beta_{\alpha}! \beta_{\alpha+1}! \dots \beta_{\alpha+r}!} \times \\ &\times \text{Per } C_{\beta_1 \dots \beta_{\alpha+r}}(U_{i_1 \dots i_r}^{(\alpha)}) \text{Per } C_{\beta_1 \dots \beta_{\alpha+r}}(V_{i_1 \dots i_r}^{(\alpha)}) + \\ &+ \sum_{\substack{\beta_1 + \dots + \beta_{\alpha+r+1} = n \\ \beta_{\alpha+r+1} \geq 1}} \frac{\lambda_1^{\beta_1} \dots \lambda_{\alpha}^{\beta_{\alpha}} \lambda_{i_1}^{\beta_{\alpha+1}} \dots \lambda_{i_{r+1}}^{\beta_{\alpha+r+1}}}{\beta_1! \dots \beta_{\alpha}! \beta_{\alpha+1}! \dots \beta_{\alpha+r+1}!} \times \\ &\times \text{Per } C_{\beta_1 \dots \beta_{\alpha+r+1}}(U_{i_1 \dots i_{r+1}}^{(\alpha)}) \text{Per } C_{\beta_1 \dots \beta_{\alpha+r+1}}(V_{i_1 \dots i_{r+1}}^{(\alpha)}). \end{aligned}$$

Thus

$$\text{Per } A_{i_1 \dots i_{r+1}}^{(\alpha)} \cong \text{Per } A_{i_1 \dots i_r}^{(\alpha)}$$

with equality if and only if $\lambda_{i_{r+1}} = 0$. Namely let us suppose that $\lambda_{i_{r+1}} > 0$. Since $\lambda_1 = 1$ according to condition (3.1) we have

$$\begin{aligned} \text{Per } C_{\beta_1=i, \beta_{i_{r+1}}=n-i}(U_{i_1 \dots i_{r+1}}^{(\alpha)}) \text{Per } C_{\beta_1=i, \beta_{i_{r+1}}=n-i}(V_{i_1 \dots i_{r+1}}^{(\alpha)}) &= 0 \\ (i = 0, 1, \dots, n-1) \end{aligned}$$

that is

$$\begin{aligned} (3.3) \quad G_i(r_1, \dots, r_n) G_i(s_1, \dots, s_n) &= 0 \\ (i = 1, \dots, n). \end{aligned}$$

Since

$$r_1 + \dots + r_n = 0, \quad r_1^2 + \dots + r_n^2 = 1,$$

we get

$$G_2(r_1, \dots, r_n) = -\frac{1}{2},$$

and similarly

$$G_2(s_1, \dots, s_n) = -\frac{1}{2},$$

i.e. condition (3.3) is not satisfied in case $i=2$. Thus $\lambda_{i_{r+1}} = 0$.

From this the proof is similar then the proof of Theorem 2.1. If i_{r+1} runs over all the numbers $\alpha+1, \dots, n$ different from $i_1, \dots, i_r$ then

$$(3.4) \quad \frac{1}{n-\alpha-r} \sum_{(i_{r+1})} \text{Per } A_{i_1 \dots i_{r+1}}^{(\alpha)} \cong \text{Per } A_{i_1 \dots i_r}^{(\alpha)}$$

with equality if and only if the numbers $\lambda_1, \dots, \lambda_n$ different from $\lambda_1, \dots, \lambda_{\alpha}, \lambda_{i_1}, \dots, \lambda_{i_r}$, are zero.

Using formula (3.4) for all combinations $i_1, \dots, i_r$ of order r of the elements $\alpha+1, \dots, n$, without repetition, and without permutation, it can be shown that $\text{Per } A_{i_1 \dots i_{r+1}}^{(\alpha)}$ appears $r+1$ times among the left hand side of these inequalities. Taking the arithmetic mean of these inequalities, i.e. dividing both side of the sum of these inequalities by $\binom{n-\alpha}{r}$, after all we divide the left hand side and the right

hand side of the sum of these inequalities by $\frac{n-\alpha-r}{r+1} \binom{n-\alpha}{r} - \binom{n-\alpha}{r+1}$ and by $\binom{n-\alpha}{r}$, respectively. Thus inequality (3.2) holds with equality if and only if $\lambda_{\alpha+1} = \dots = \lambda_n = 0$.

Corollary 3.1. *Let $A = U \wedge V^*$ be the polar representation of $A \in K$. If condition (3.1) is satisfied then*

$$G(A_{r+1}^{(1)}) \cong G(A_r^{(1)}) \cong \frac{n!}{n^n}$$

$$(r = 0, 1, \dots, n-2)$$

with equality if and only if $A = S_0$.

PROOF. In this case $G_0^{(1)}(A) = u_1 v_1^* = S_0$ with equality if and only if $\lambda_\alpha = \dots = \lambda_n = 0$, i.e. $A = S_0$.

Corollary 3.2. *If the matrix $A \in K$ satisfies condition (3.1) then*

$$G_{r+1}^{(0)}(A) > G_r^{(0)}(A) \quad (r = 0, 1, \dots, n-1).$$

PROOF. Equality holds if and only if $\lambda_1 = \dots = \lambda_n = 0$, contradicting to $\lambda_1 = 1$.

Corollary 3.3. *If $A \in K$ is a symmetric positive semidefinite matrix then*

$$G_{r+1}^{(\alpha)}(A) \cong G_r^{(\alpha)}(A)$$

$$(\alpha = 0, 1, \dots, n-1; r = 0, 1, \dots, n-\alpha-1; r+\alpha \cong 1).$$

In particular

$$G_{r+1}^{(1)}(A) \cong G_r^{(1)}(A) \cong \frac{n!}{n^n} \quad (r = 0, 1, \dots, n-2)$$

with equality if and only if $A = S_0$. Moreover

$$G_{r+1}^{(0)}(A) > G_r^{(0)}(A) \quad (r = 0, 1, \dots, n-2).$$

PROOF. In this case condition (3.1) is satisfied trivially.

Since $G_{n-1}^{(1)}(A) = A$, Corollary 3.3. contains Theorem 1.2. too.

Finally the importance of Theorem 3.1. is expressed by the following theorem:

Theorem 3.2. *Condition (3.1) is satisfied by the elements of an infinite subset of K , containing non-symmetric non-positive semidefinite matrices.*

PROOF. Let us consider the set of the $n \times n$ orthogonal matrices, which have $1/\sqrt{n}$ as the elements of their first column. The power of these orthogonal matrices is infinite. We decompose now this set into mutually disjoint subsets. Two orthogonal matrices U and V belong to the same subset if condition (3.1) is satisfied by them.

Since we have finitely many subsets (their numbers is equal to $\exp_2 \binom{2n-1}{2}$), at least one subset should contain infinitely many elements. This completes the proof of Theorem 3.2.

References

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