

Localization in duo rings

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1. *Introduction.* A duo ring is a ring in which every one sided ideal is two sided. Such rings are called two-sided rings in [8] and [9]. Duo rings were so named by FELLER in [2]. Several examples of noncommutative duo rings were given in [4]. In [1], SCHILLING first studied noncommutative valuation rings and lemma 2 of [1] shows that they are duo rings. Since every noncommutative totally ordered group is the value group of some valuation [5], the collection of noncommutative valuation rings is a large class of duo rings.

It is easy to see that R is a duo ring if and only if $aR = Ra$ for all $a \in R$. Thus when R is an integral domain, R has a (left and right) division ring of quotients $D = \{a^{-1}b \mid a, b \in R, a \neq 0\} = \{ba^{-1} \mid a, b \in R, a \neq 0\}$. In [1], Schilling showed that if R is valuation ring with unique maximal ideal P then $x^{-1}Px = P$ and $x^{-1}Rx = R$ for all $x \in D^* = D \setminus \{0\}$. P and R are said to be invariant under inner automorphisms of D^* . Schilling used the invariance of R and P to construct a quotient ring R_P with properties similar to those in the case where R is commutative.

In this paper we extend Schillings ideas to integral domains which are duo rings.

Let R be a duo ring with identity $1 \neq 0$. A subset S of R is called a multiplicative system in R if $0 \notin S$ and $s_1s_2 \in S$ for all $s_1, s_2 \in S$. The following theorem shows that prime ideals in duo rings have the same characterization as prime ideals in commutative rings. Thus let P be an ideal of R .

Theorem. *The following are equivalent in R .*

- (i) P is a prime ideal of R .
- (ii) For $a, b \in R$, if $ab \in P$, then $a \in P$ or $b \in P$.
- (iii) $R \setminus P$ is a multiplicative system in R .

PROOF. (i) $\Leftrightarrow$ (ii) is found in [3]. (ii) $\Leftrightarrow$ (iii) is clear.

In this paper all rings are integral domains with identity $1 \neq 0$. Section 2 characterizes duo rings in terms of groups of divisibility. Section 3 treats the problem of localization with respect to multiplicative systems.

2. Groups of divisibility in integral domains which are duo rings.

Let R be an integral domain which is a duo ring and let D denote the division ring of quotients of R . U denotes the multiplicative group of units of R and D^* denotes the multiplicative group of nonzero elements of D . This section characterizes duo domains in terms of groups of divisibility.

Theorem 2.1. U is a normal subgroup of D^* .

PROOF. Let $x \in D^*$, $u \in U$. Then $x = a^{-1}b = ca^{-1}$ for some $a, b, c \in R$. Thus $xux^{-1} = a^{-1}bub^{-1}a = ca^{-1}uac^{-1}$. So it is sufficient to show that $a^{-1}ua \in U$, $aua^{-1} \in U$ for all $a \in R$. So let $a \in R$, $a \neq 0$, and let $u \in U$. Then $a^{-1}ua = a^{-1}ar = r \in R$, where $u^{-1}a = as$. So $(a^{-1}ua)(a^{-1}u^{-1}a) = 1 = rs = sr$. Thus $r = a^{-1}ua \in U$. Similarly, $aua^{-1} \in U$ for all $a \in R$, $a \neq 0$ and U is normal in D^* .

Definition 2.2. D^*/U is called the group of divisibility of R .

Proposition 2.3. D^*/U is a partially ordered group where $xU \leq yU \Leftrightarrow x^{-1}y \in R$.

The proof is straight forward and is omitted.

Let $v: D^* \rightarrow D^*/U$ denote the natural map. Then

$$(1) v(xy) = xyU = (xU)(yU) = v(x)v(y).$$

(2) If $x, y \in D^*$ and $x + y \neq 0$, then $v(x + y) \leq v(t)$ for any $t \in D^*$ such that $v(t) \leq v(x)$ and $v(t) \leq v(y)$.

For suppose $v(t) \leq v(x)$ and $v(t) \leq v(y)$. Then $t^{-1}x, t^{-1}y \in R$ and so $t^{-1}x + t^{-1}y = t^{-1}(x + y) \in R$. Then $tU \leq (x + y)U$, i.e., $v(t) \leq v(x + y)$.

Definition 2.4 [6, pg. 10]. A partially ordered (p.o.) group is said to be directed if any pair $a, b \in G$ has a lower bound (l.b.) $c \in G$ (equivalently, if any pair $a, b \in G$ has an upper bound).

Proposition 2.5. With R, D, D^*, U as above, D^*/U is a directed group.

PROOF. Let $xU, yU \in D^*/U$. Then $x = a^{-1}b, y = c^{-1}d$ for some $a, b, c, d \in R$. Then $a^{-1}U \leq xU$ and $c^{-1}U \leq yU$. This gives $(ac)^{-1}U \leq a^{-1}U$ and $(ac)^{-1}U \leq c^{-1}U$. For $aca^{-1} = c_0aa^{-1} = c_0$, where $ac = c_0a$ for some $c_0 \in R$, and $acc^{-1} = a \in R$. Thus $(ac)^{-1}U$ is a lower bound for xU and yU and D^*/U is directed.

Now, let D be a division ring and let G be a directed group. Let $v: D \rightarrow G \cup \{\infty\}$ be a function satisfying (1), (2) of Proposition 2.3 and (3) $v(0) = \infty$ and v maps D onto $G \cup \{\infty\}$. Let $R = \{x \in D \mid v(x) \leq e\}$. The following extends Schilling [1] and YaKabe [7]. We assume without loss of generality that G is generated by $U(e) = \{x \in G \mid x \leq e\}$ [6].

Proposition 2.6. (i) R is a subring of D ; (ii) R is a duo ring; (iii) D is the ring of quotients of R .

PROOF. (i) Clearly $v(1) = e$. Since $(-1)^2 = 1$, this gives $v(-1) = v(-1)^{-1}$. Let $a \in G$ be a lower bound for $v(-1) = v(-1)^{-1}$ and e . Then $a \leq v(-1)$ and $a \leq e$. This gives $a \leq v(-1) = v(-1)^{-1} \leq a^{-1} \leq e$ so $v(-1) \leq e$ and $v(-1) = v(-1)^{-1} \leq e$, giving $v(-1) = e$. It follows that $v(x) = v(-x)$ for all $x \in D$, and that R is a subring of D . Let $a \in R$, and let $A = \{x \in R \mid v(x) \leq v(a)\}$. Clearly $aR = A = Ra$ and R is duo. That D is the ring of quotients of R follows from the fact that v maps D onto $G \cup \{\infty\}$.

We combine the above to get the following.

Theorem 2.7. Let D be a division ring. A subring R of D is a duo ring with D as ring of quotients if and only if there is a directed group $(G, \leq)$ and a map from D onto $G \cup \{\infty\}$ satisfying properties (1), (2), (3).

3. Multiplicative systems and localization in duo domains.

Let R be a duo domain with D as division ring of quotients. Let T be a multiplicative system in R . The collection $\mathcal{S}$ of ideals A of R such that $A \cap T = \emptyset$ is not empty and Zorn's lemma shows that $\mathcal{S}$ contains maximal elements. The argument that the maximal elements of $\mathcal{S}$ are prime ideals of R is similar to the commutative case using the fact that $Ra = aR$ for all $a \in R$. Let P_λ denote the collection of maximal elements of $\mathcal{S}$, and let $S = R \setminus (\bigcup_{\lambda} P_\lambda) = \bigcap_{\lambda} (R \setminus P_\lambda)$. Then S is a multiplicative system in R and $T \subset S$. When R is commutative, S is called the saturation of T and $R_T = R_S$ (see [10], pg. 13).

Lemma 3.1. *Let S be a saturated multiplicative system in R , and let $s_1, s_2 \in S$. Then $s_1 s_2 = s_2 s_1 = s_2 s$, for some $s_1, s_2 \in S$.*

PROOF. $s_1 s_2 = s_2 s'_1$ for some $s'_1 \in R$. If $s'_1 \in S = \bigcap_{\lambda} (R \setminus P_\lambda)$ then $s'_1 \in P_\lambda$ for some λ and then $s_1 s_2 \in S \cap P_\lambda = \emptyset$, a contradiction. So $s'_1 \in S$. Similarly $s'_2 \in S$.

We observe that the above lemma holds when R is not an integral domain.

Proposition 3.2. *Let S be a saturated multiplicative system in R . Then R_S is a subring of D , where $R_S = \{s^{-1}a_s | s \in S, a_s \in R\}$.*

PROOF. It is straight forward to show that R_S is closed under multiplication. To see that R_S is closed under addition, let $s^{-1}a, t^{-1}b \in R_S$. Then $s^{-1}a + t^{-1}b = (s't)^{-1}(ta + sb)$, where $ts = s't$. The above lemma show that $s' \in S$.

Definition 3.3. If T is a multiplicative system in R , we define $R_T = R_S$, where S is the saturation of T .

In the case where R is a noncommutative valuation ring with unique maximal ideal P , SCHILLING [1] showed that P is invariant under all inner automorphisms, i.e., that $xPx^{-1} = P$ for all $x \in D^*$. It follows that in a duo domain R , a prime ideal P is invariant if and only if $S = R \setminus P$ is invariant.

Definition 3.4. Let T be a saturated multiplicative system in R . T is said to be invariant iff $xTx^{-1} = T$ for all $x \in D^*$.

The following characterizes invariant multiplicative systems.

Theorem 3.5. *Let $T = \bigcap_{\lambda} (R \setminus P_\lambda)$ be a saturated multiplicative system in R , where $\{P_\lambda\} = \mathcal{S}$ is a collection of prime ideals of R . Then T is invariant iff $xP_\lambda x^{-1} \in \mathcal{S}$ for all $x \in D^*$, $P_\lambda \in \mathcal{S}$.*

PROOF. It is easy to see that if $x \in D^*$, $P_\lambda \in \mathcal{S}$, then $xP_\lambda x^{-1}$ is a prime ideal of R .

Suppose that $\mathcal{S}$ is closed under inner automorphisms. Let $t \in T, x \in D^*$. If $xtx^{-1} \notin T$, then $xtx^{-1} \in P_\lambda$ for some λ . Say $xtx^{-1} = a \in P_\lambda$. Then $t = x^{-1}ax \in x^{-1}P_\lambda x \in \mathcal{S}$. Then $t \in T \cap (\bigcup_{\lambda} P_\lambda) = [R \setminus (\bigcup_{\lambda} P_\lambda)] \cap (\bigcup_{\lambda} P_\lambda) = \emptyset$, a contradiction. So $xTx^{-1} \subseteq T$ for all $x \in D^*$. It follows that $xTx^{-1} = T$ for all $x \in D^*$.

On the other hand, if T is invariant, then $xtx^{-1} \in T$ for all $x \in D^*, t \in T$. So $T = xTx^{-1} \cap P_\lambda = \emptyset$ for all $P_\lambda \in \mathcal{S}$. Thus $T \cap x^{-1}P_\lambda x = \emptyset$ for all $P_\lambda \in \mathcal{S}$ and $\mathcal{S}$ is closed under inner automorphisms.

In general, if T is a saturated multiplicative system in R , it is not always true that R_T is a duo domain. The following gives a sufficient condition for R_T to be a duo domain.

Proposition 3.6. *Let T be a saturated multiplicative system in R . If T is invariant, then R_T is a duo ring.*

PROOF. Suppose T is invariant. Then for $t \in T$, $a \in R^*$, we have $a^{-1}ta = t' \in T$, so $ta = at_1$, and $ata^{-1} = t_2 \in T$, i.e., $at = t_2a$. Let $x = s^{-1}b$, $y = t^{-1}c$ be arbitrary nonzero elements of R_M . Then $xy = (s^{-1}b)(t^{-1}c) = s^{-1}(bt^{-1})c$. Since T is invariant, $bt^{-1} = t_1^{-1}b$ for some $t_1 \in T$. Then $xy = s^{-1}(t_1^{-1}b)c$. Now, $bc = c'b$ for some $c' \in R$ and since M is saturated, $s^{-1}t_1^{-1} = t_2^{-1}s^{-1}$ for some $t_2 \in T$. Then $xy = t_2^{-1}s^{-1}c'b$. We have $s^{-1}c' = c''s^{-1}$ for some $c'' \in R$, so $xy = (t_2^{-1}c'')s^{-1}b = y'x$, where $y' = t_2^{-1}c$. Thus $xR_T \subseteq R_Tx$. Similarly $R_Tx \subseteq xR_T$ and R_T is a duo ring.

Lemma 3.7. (1) *If $\{T_\lambda\}$ is a collection of subsets of R then for any $x \in D^*$, $x^{-1}(\cap T_\lambda)x = \cap x^{-1}T_\lambda x$.*

(2) *If S is a multiplicative system in R , then for any $x \in D^*$, $x^{-1}Sx$ is a multiplicative system in R and $x^{-1}R_Sx = R_{x^{-1}Sx}$.*

PROOF. (1) is straight forward. It is also easy to show that $x^{-1}Sx$ is a multiplicative system if S is a multiplicative system. So let $y \in x^{-1}R_Sx$. Then $y = x^{-1}(s^{-1}r)x = (x^{-1}s^{-1}x)(x^{-1}rx) = (x^{-1}sx)^{-1}(x^{-1}rx) \in R_{x^{-1}Sx}$ since $x^{-1}rx \in R$. On the other hand, let $Z \in R_{x^{-1}Sx}$, say $Z = (x^{-1}sx)^{-1}r$. Since $R = x^{-1}Rx$, we have $r = x^{-1}tx$ for some $t \in R$. Then $Z = (x^{-1}sx)^{-1}r = (x^{-1}s^{-1}x)x^{-1}tx = x^{-1}(s^{-1}t)x \in x^{-1}R_Sx$, and we have equality.

Now, let S be a saturated multiplicative system in R , say $S = \cap \{R - P \mid P \in \mathcal{P}\}$, where $\mathcal{P}$ is the collection of prime ideals of R which are maximal with respect to the property that $P \cap S = \emptyset$. If R_S is a duo ring, then $R_S = x^{-1}R_Sx = R_{x^{-1}Sx} = R_{\cap x^{-1}(R \setminus P)x}$. Thus $\mathcal{P}$ is closed under inner automorphisms and S is invariant by 3.5 above. This together with 3.7 above gives

Theorem 3.8. *Let S be a saturated multiplicative system in R . Then R_S is a duo ring $\iff S$ is invariant.*

Let S be a saturated multiplicative system in R . $R_S = \{s^{-1}r \mid s \in S, r \in R\}$. It can be easily shown that $R_S = \{s^{-1}r \mid s \in S, r \in R\} = \{rs^{-1} \mid s \in S, r \in R\}$. Let A be an ideal of R . The right extension of A to R_S is

$$A_r^e = AR_S = \left\{ \sum a_i r_i s_i^{-1} \mid a_i \in A, r_i \in R, s_i \in S \right\} = \{as^{-1} \mid a \in A, s \in S\}.$$

Similarly, the left extension of A to R_S is

$$A_l^e = R_S A = \left\{ \sum s_i^{-1} r_i a_i \mid s_i \in S, r_i \in R, a_i \in A \right\} = \{s^{-1}a \mid s \in S, a \in A\}.$$

Proposition 3.9. (1) *The right extension of A to R_S is a right ideal of R_S*
(2) *The left extension of A to R_S is a left ideal of R_S .*

PROOF. (1) Suppose $as^{-1} \in A_r^e$ and $rt^{-1} \in R_S$. Then $(as^{-1})(rt^{-1}) = a(s^{-1}rs)s^{-1}t^{-1}$. Then $s^{-1}rs \in R$, so $a(s^{-1}rs) = a' \in A$, and so $(as^{-1})(rt^{-1}) = a'(ts)^{-1} \in A_r^e$. Similarly, $R_S A$ is a left ideal of R_S .

Definition 3.10. Let A be an ideal of R , S a saturated multiplicative system in R . A is said to be S -invariant if $s^{-1}As = A$ for all $s \in S$.

Clearly, if A is invariant, then A is S -invariant for any S .

Let P be any prime ideal of R , and let $S(P) = R \setminus P$. Then P is $S(P)$ -invariant. For let $s \in S = S(P)$ and $a \in P$. Then if $s^{-1}as = t \in S$, then $as = st \in P$ and $s \notin P$, $t \notin P$ a contradiction. Thus $s^{-1}Ps \subseteq P$, and P is $S(P)$ -invariant.

Proposition 3.11. Let A be an ideal of R , S a saturated multiplicative system in R , with $A \cap S = \emptyset$. If A is S -invariant, then $AR_S = R_S A$ is an ideal of R .

PROOF. Let $as^{-1} \in AR_S$. Then $as^{-1} = s^{-1}(sas^{-1})$ and $sas^{-1} \in A$ since A is S -invariant. Thus $AR_S \subseteq R_S A$. Similarly $R_S A \subseteq AR_S$.

It is clear that AR_S is a proper right ideal (1.8) of $R_S \Leftrightarrow A \cap S = \emptyset$; and similarly for $R_S A$.

As is the case when R is commutative, if P is a prime ideal of R and $S = R \setminus P$, we write R_P for R_S .

Corollary 3.12. For any prime ideal P of R , R_P is a local ring with unique maximal ideal PR_P .

Now, let A, B be ideals of R .

Definition [13, pg. 254]. $A \cdot B = \{d \in R \mid Bd \subseteq A\}$. $A \cdot B$ is an ideal of R [13].

Let $\{M_\lambda\}$ denote the collection of maximal ideals of R . The next lemma is similar to [12, pg. 94].

Lemma 3.13. Let A be an ideal of R , and let $x \in R$. If $x \in AR_{M_\lambda}$ for all λ , then $x \in A$.

PROOF. If $x \in AR_{M_\lambda}$ then $x = at_m^{-1}$, where $a \in A$, $t_m \in R \setminus M_\lambda$. Then $xt_m = a \in A$, and so $A R_x \not\subseteq M_\lambda$ since $t_m \in R \setminus M_\lambda$. Thus $A \cdot R_x$ is contained in no maximal ideal of R and $A \cdot R_x = R$. Thus $1 \in A \cdot R_x$ and $x \in A$.

Corollary 3.14. $R = \bigcap_{\lambda} R_{M_\lambda}$.

PROOF. Clearly $R \subseteq \bigcap_{\lambda} R_{M_\lambda}$. Let $z = y^{-1}x \in \bigcap_{\lambda} R_{M_\lambda}$ for some $x, y \in R$. Then $x \in (Ry)R_{M_\lambda} = yR$ for all λ . Thus $x \in Ry = yR$, and $y^{-1}x = z \in R$, and $R = \bigcap_{\lambda} R_{M_\lambda}$.

Now, let S be an invariant multiplicative system in R so that R_S is a duo ring. Let A be an ideal of R and let $\mathcal{A}$ be an ideal of R_S .

Definition 3.15 [11, pg. 218].

(i) The extension of A to R_S is $A^e = AR_S = R_S A$.

(ii) The contraction of $\mathcal{A}$ to R is $\mathcal{A}^c = \mathcal{A} \cap R$.

Then as in [11, pg. 219] we have the following

Lemma 3.16. Let $\mathcal{A}, \mathcal{B}$ be ideals of R_S , A, B ideals of R .

- (1) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^c \subseteq \mathcal{B}^c$; $A \subseteq B \Rightarrow A^e \subseteq B^e$
- (2) $\mathcal{A}^{cc} \subseteq \mathcal{A}$; $A^{ee} \supseteq A$
- (3) $\mathcal{A}^{cec} = \mathcal{A}^c$; $A^{eec} = A^e$

Corollary 3.17. [11, p. 223]. Let $\mathcal{A}$ be an ideal of R_S . Then $\mathcal{A} = \mathcal{A}^{ce}$, i.e., every ideal of R_S is an extended ideal of R .

PROOF. Let $a' \in \mathcal{A}$. Then $a' = s^{-1}$, for some $s \in S$, $r \in R$. Then $r \in \mathcal{A}^c$, and $a' \in \mathcal{A}^{ce}$, and $\mathcal{A} \subseteq \mathcal{A}^{ce}$. Clearly $\mathcal{A}^{ce} \subseteq \mathcal{A}$.

It is clear that (i) if $\mathcal{A}$ is an invariant ideal of R , then $\mathcal{A}^c = \mathcal{A}R_S$ is an invariant ideal of R_S ; and (ii) if $\mathcal{A}$ is an invariant ideal of R_S , then $\mathcal{A}^c = \mathcal{A} \cap R$ is an invariant ideal of R .

Theorem 3.18. Let $\mathcal{P}$ be an invariant prime ideal of R_S . Then $\mathcal{P} = PR_S$ where $P = \mathcal{P} \cap R = \mathcal{P}^c$ is an invariant prime of R and $R_P = (R_S)_{PR_S}$.

PROOF. Let $T = R \setminus P$. Then $S \subseteq T$ since $P \cap S = \emptyset$. Let $x \in R_P$. Then $x = t^{-1}r$, $t \in T$, $r \in R$. We have $\mathcal{P} = PR_S$ by corollary 3.17 above, and so $PR_S = \{s^{-1}a \mid s \in S, a \in P\}$. Then $R_S \setminus PR_S = \{s^{-1}t \mid s \in S, t \in P = \{s^{-1}t \mid s \in S, t \in T\}$. If $x \in (R_S)_{PR_S}$, then $x = (s^{-1}t)^{-1}(s_1^{-1}r)$, where $t \in T$, $s_1 s_1^{-1} \in S$. Then $x = t^{-1}ss_1^{-1}r = t^{-1}s_1^{-1}(s_1ss_1^{-1})r = (s_1t)^{-1}(s''r) \in R_P$. So $(R_S)_{PR_S} \subseteq R_P$. On the other hand, $T \subseteq R_S - PR_S$ and $R \subseteq R_S$, so $R_P = R_T \subseteq (R_S)_{PR_S}$.

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