

Extendibility of $*$ -representations from $*$ -ideals of $*$ -semigroups

By ZOLTÁN SEBESTYÉN (Budapest)

In a previous paper [3] the author proved a result concerning extendibility of a $*$ -representation (on Hilbert space) of a $*$ -ideal to the whole $*$ -algebra under a relatively simple condition known from the same problem with respect to C^* -seminorms [4].

The aim of this note is to extend the extendibility question to a more general case. First we shall prove that a $*$ -representation of a $*$ -ideal of a $*$ -semigroup is extendible to the whole semigroup if (and only if) the preceding condition ((1) below) holds. The case when our $*$ -semigroup is embedded in a Banach $*$ -algebra the extendibility problem is known from the author's investigations concerning moment type problems [5].

Let G be a (multiplicative, associative) semigroup with involution $(*)$, briefly a $*$ -semigroup, introduced by SZ.-NAGY [2] concerning dilation problems. A sub-semigroup J of G is a $*$ -ideal if it is closed under the involution and $GJ \subset G$ (hence $JG \subset G$). A $*$ -homomorphism of J into $B(H)$, the C^* -algebra of all bounded linear operators on a (complex) Hilbert space H , is called a $*$ -representation of J on H .

Lemma. *Let T be a $*$ -representation of J on the Hilbert space H and let $H_0 = \{x \in H : T_h x = 0 \text{ for each } h \in J\}$. Each $x \in H$ orthogonal to H_0 belongs to the closed subspace in H spanned by $\{T_h x : h \in J\}$, which is orthogonal to H_0 .*

PROOF. For a fixed $0 \neq x \in H_0^\perp$, where $H_0^\perp$ denote the orthogonal complement to H_0 in H , let

$$H_x := \vee \{T_h x : h \in J\}$$

be the closed subspace of H spanned by elements $T_h x, h$ running through J . If y denotes the unique element of H_x with the property that $x - y$ is orthogonal to H_x we have to show $x = y$. First of all $x - y \in H_0$ because

$$\|T_h(x - y)\|^2 = (T_{h^*h}(x - y), x - y) = -(T_{h^*h}y, x - y) = 0$$

holds for any h in J since $y \in H_x$ immediately implies $T_{h^*h}y \in H_x$. Hence we have that

$$\|x - y\|^2 = (x - y, x - y) = (x, x - y) = 0$$

indeed (because x is orthogonal to H_0 , hence to $x - y$). The proof of Lemma is complete.

Theorem 1. *Let T be a $*$ -representation of the $*$ -ideal J of the $*$ -semigroup G on a Hilbert space H . There exists a $*$ -representation S of G on H extending T if and only if*

$$(1) \quad p(g) := \inf \{M > 0: \|T_{gh}\| \leq M\|T_h\| (h \in J)\} < \infty \quad (g \in G)$$

PROOF. Assuming such an S the condition (1) is obviously satisfied because of $gh \in J$ provided $g \in G, h \in J$ and because

$$\|T_{gh}\| = \|S_{gh}\| \leq \|S_g\| \|S_h\| = \|S_g\| \|T_h\|.$$

To prove the sufficiency of (1) we take first S to be zero on $H_0 := \{x \in H: T_h x = 0 \text{ for each } h \in J\}$, a closed subspace of H on which T is also trivial. For if $x \in H_0^\perp$, i.e. x is orthogonal to H_0 , we know from the Lemma that x belongs to $H_x := \bigvee \{T_h x: h \in J\}$, the closed subspace spanned by $T_h x$'s as h runs through J . It is obvious that for any x in $H_0^\perp$ which is of the form $\sum_h T_h x_h$ (finite sum) with $h \in J, x_h \in H$, we have to define $S_g (g \in G)$ as follows

$$(2) \quad S_g x := \sum_g T_{gh} x_h \quad \text{for any } g \text{ in } G.$$

Since such elements are dense in $H_0^\perp$ by Lemma, we have only to prove that (2) is correct for S and gives a bounded operator on $H_0^\perp$. The remainder of the proof is then clear, namely that S establishes a $*$ -representation of G on H . But we have

$$\left\| \sum_h T_{gh} x_h \right\|^2 = \left(\sum_h T_{g^*gh} x_h, \sum_h T_h x_h \right) \leq \left\| \sum_h T_{g^*gh} x_h \right\| \left\| \sum_h T_h x_h \right\|$$

so that by induction we have also that

$$\begin{aligned} \left\| \sum_h T_{gh} x_h \right\|^{2^{n+1}} &\leq \left\| \sum_h T_{(g^*g)^{2^n}} x_h \right\| \left\| \sum_h T_h x_h \right\|^{2^{n+1}-1} \leq \\ &\leq \sum_h \|T_{(g^*g)^{2^n} h}\| \|x_h\| \left\| \sum_h T_h x_h \right\|^{2^{n+1}-1} \leq \\ (\text{by (1)}) \quad &\leq \sum_h p((g^*g)^{2^n}) \|T_h\| \|x_h\| \left\| \sum_h T_h x_h \right\|^{2^{n+1}-1} \quad (n = 0, 1, 2, \dots) \end{aligned}$$

and hence that

$$(3) \quad \left\| \sum_h T_{gh} x_h \right\| \leq \lim_{n \rightarrow \infty} p((g^*g)^{2^n})^{2^{-n-1}} \left\| \sum_h T_h x_h \right\|.$$

We need properties for p as follows

$$(4) \quad p(gg_1) \leq p(g)p(g_1) \quad (g, g_1 \in G)$$

$$(5) \quad p(g^*g) = p(g)^2 \quad (g \in G)$$

The proof of (4) follows by the definition of p since

$$\|T_{gg_1 h}\| \leq p(g)\|T_{g_1 h}\| \leq p(g)p(g_1)\|T_h\|$$

holds for each $g, g_1 \in G, h \in J$. But the estimates

$$\|T_{gh}\|^2 = \|T_{h^*g^*gh}\| \leq \|T_{h^*}\| \|T_{g^*gh}\| \leq p(g^*g) \|T_h\|^2$$

for $g \in G, h \in J$ imply

$$p(g)^2 \leq p(g^*g) \leq p(g^*)p(g),$$

$$p(g) \leq p(g^*) \leq p((g^*)^*) = p(g)$$

for each $g \in G$ and hence (5) since thus

$$p(g)^2 \leq p(g^*g) \leq p(g)^2$$

holds also for any $g \in G$.

To end the proof we have by (2), (3), (4), (5) that

$$\left\| \sum_h T_{gh} x_h \right\| \leq p(g) \left\| \sum_h T_h x_h \right\|$$

hence that

$$\|S_g\| \leq p(g)$$

holds for each $g \in G$ indeed.

Theorem 2. Let G be a *-semigroup in a Banach *-algebra A such that G generates a norm dense *-subalgebra in A . Let further T be a *-representation of a *-ideal J of G on a Hilbert space H . There exists a *-representation S of A extending T on the same space H if and only if

$$(6) \quad M := \sup \left\{ \left\| \sum_g c_g T_{gh} \right\| : h \in J, \|T_h\| \left\| \sum_g c_g g \right\| \leq 1 \right\} < \infty$$

holds, where $\{C_g\}$ is any finite set of complex numbers indexed by elements of G .

PROOF. The necessity of the condition (6) is obvious since any *-representation of a Banach *-algebra on Hilbert space is automatically continuous (see [1]).

We shall prove that (6) ensures the desired *-representation S on H . Since (6) clearly implies (1) we have by Theorem 1 a *-representation S of G on H such that

$$(7) \quad \|S_g\| \leq M \|g\| \quad (g \in G)$$

We need an extension to (7) for the *-subalgebra $A(G)$ in A generated by G as follows

$$(8) \quad \|S_a\| \leq C \|a\| \quad (a \in A(G))$$

where for an $a = \sum_g c_g g \in A(G)$ the representing operator on H is S_a and $C > 0$ is a constant depending only on A .

$$(9) \quad S_a = \sum_g c_g S_g \quad (a = \sum_g c_g g \in A(G))$$

is thus well defined. To prove (8) we shall use the same argument as before: for $x = \sum_h T_h x_h$ ($h \in J$, $x_h \in H$)

$$\begin{aligned} \|S_a x\|^2 &= (S_a^* a x, x) \leq \|S_a^* a x\| \|x\|, \\ \|S_a x\|^{2^{n+1}} &\leq \|S_{(a^* a)^{2^n}} x\| \|x\|^{2^{n+1}-1} \leq \sum_h \|S_{(a^* a)^{2^n}} T_h x_h\| \|x\|^{2^{n+1}-1} = \\ &= \sum_h \left\| \sum_s \lambda_s T_{g_s h} x_h \right\| \|x\|^{2^{n+1}-1} \leq \sum_h M \left\| \sum_s \lambda_s g_s \right\| \|T_h\| \|x_h\| \|x\|^{2^{n+2}-1} = \\ &= M \|(a^* a)^{2^n}\| \|x\|^{2^{n+1}-1} \sum_h \|T_h\| \|x_h\| \end{aligned}$$

and thus also that

$$\|S_a x\| \leq \lim \| (a^* a)^{2^n} \|^{2^{-n-1}} \|x\| = r(a^* a)^{1/2} \|x\|,$$

where $r(a^* a)$ denotes the spectral radius of $a^* a$ in A , and $\sum_s \lambda_s g_s$ stands for $(a^* a)^{2^n}$.

By the way

$$(10) \quad r(a^* a)^{1/2} \leq c \|a\| \quad (a \in A)$$

holds true, as is known for Banach *-algebras from [1, p. 196], implying (8) indeed.

Since we have thus really a *-representation S of $A(G)$ on H which is continuous with respect to the norm of A , the norm denseness of $A(G)$ in A implies the statement of the theorem; only (unique) continuous extension of S is needed to A . The proof is complete.

References

- [1] F. F. BONSALL—J. DUNCAN, Complete Normed Algebras, Erg. Math. Band 80, Springer (Berlin—Heidelberg—New York), 1973.
- [2] F. RIESZ—B. SZ.-NAGY, Functional Analysis, New York, 1960.
- [3] Z. SEBESTYÉN, On extendibility of *-representations from *-ideals, *Acta Sci. Math. (Szeged)*, **40** (1978) 169—174.
- [4] Z. SEBESTYÉN, Every C^* -seminorm is automatically submultiplicative, *Periodica Math. Hungarica*, **10** (1979), 1—8.
- [5] Z. SEBESTYÉN, Moment problem for dilatable semigroup of operators, *Acta Sci. Math. (Szeged)*, **45** (1983), 365—376.

DEPARTMENT OF MATH. ANALYSIS II.
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÜT 6—8.
1088 BUDAPEST, HUNGARY

(Received July 26, 1982.)